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$*$-U-REGULAR RING

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1. Preliminary definitions and results

DEFINITIONS 1. A ring R with unity 1 is called *U-regular* if for every a in R there exists a unit u in R such that $aua=a$

2. A *$*$-ring* is a ring with an involution $x \longrightarrow x^*$:

$$(x^*)^*=x,\ (x+y)^*=x^*+y^*,\ (xy)^*=y^*x^*.$$

3. An element e in R is called *projection* if $e=e^*$ and $e^2=e$.
4. The involution of a $*$-ring is said to be *proper* if $a^*a=0 \Longrightarrow a=0$.
5. An element a in R such that $aa^*a=a$ is called *partial isometry*.
6. A *$*$-U-regular ring* is a U-regular ring with proper involution.
7. An idempotent e is called normal if $e^*e=ee^*$.

The main purpose of this paper is to prove the theorem:

"A Commutative U--regular ring R with a suitable involution is a $*$-U-regular ring if $ab=0$ and $aua=ava$, a,b, units u,v in R."

Thus we make the conditions that R is $*$-ring and $*$-regular, superfluous, assumed in proposition 3 [1, pp. 229], which is as follows:

If R is a $*$-ring with unity, the following conditions are equivalent:

(i) R is $*$-regular

(ii) for each $x \in R$, there exists a projection e such that $Rx=Re$.

(iii) R is regular and is a Rickart $*$-ring.

The notations and conventions are as in Sterling K-Berberian [1].

We shall prove few simple results, R will denote commutative U-regular ring.

R_1. $aua=ava$, u,v units in R, then $au=av$.

The proof is trivial.

R_2. $(a+1-au)$ is invertible in R.

PROOF. There are units w in R such that $(a+1-au)w(a+1-au)=a+1-au$. Multiplying the above equations by au and $(1-au)$ we get $awa=a$ and $w(1-au)=1-au$ respectively. From R_1 we get $au=aw$. Also $w(a+1-au)=wa+1-au=1$.

Define $*:R \longrightarrow R$ by $a \longrightarrow a^*$, where $a^*=au(a+1-au)^{-1}$, $aua=a$.

The mapping is well defined from R_1 and R_2.

R_3. $a^*au=a^*$, since au is idempotent.

R_4. $aa^*a=a$, since $a^*(a+1-au)=au$ and so $a^*a=au$. Hence the result.

R_5. $a^*=au(a+1-au)^{-1}=a^*a(a+1-a^*a)^{-1}$(from R_4).

R_6. $a^*v=au$ *if* $a^*va^*=a^*$. since $a^*v=(a^*au)v=(a^*a^*a)v=(a^*va^*)a=a^*a=au$ (from R_3 and R_4).

From the above results we deduce

(i) $a^{**}=a$ (ii) $(a+b)^*=a^*+b^*$ if $ab=0$, $aua=ava=a$

(iii) $(ab)^*=a^*b^*$ (iv) $a^*a=0\Longrightarrow a=0$.

R is U-regular, so $a^*va=a^*, a^*\in R$.

$a(a^*+1-a^*v)=a(a^*+1-au)$ (from R_6), $aa^*=au$ (from R_4)$=a^*v$(from R_6)

Hence $a=a^*v(a^*+1-a^*v)^{-1}=(a^*)^*$.

For (ii):

Since $ab=0$ implies $a^*b=0=ab^*$, from R_4 we get

$$(a+b)(a+b)^*(a+b)=(a+b)(a^*+b^*)(a+b)$$

So R_1 implies $(a+b)(a+b)^*=(a+b)(a^*+b^*)=aa^*+bb^*$

$$=aa^*(b+1-bb^*)+bb^*(a+1-aa^*) \qquad (A)$$

Also $(a+b)+1-(a+b)(a+b)^*=(b+1-bb^*)(a+1-aa^*)$ (B)

From (A) and (B) we get $(a+b)^*=a^*+b^*$.

For (iii):

$$(ab)(a^*b^*)(ab)=(aa^*a)(bb^*b)=ab=ab(ab)^*(ab)$$

Hence $ab\ a^*b^*=ab(ab)^*$ from R_1. $=(ab)^*\{ab+1-(ab)(ab)^*\}$ from (R_5)

Now $ab=(ab)(ab)^*(ab)=(ab)(ab)^*\{ab+1-(ab)(ab)^*\}=ab\ a^*b^*\{ab+1-(ab)(ab)^*\}$

$$=ab(ab)^*(a+1-aa^*)(b+1-bb^*)$$

So $(ab)(a^*b^*)\{ab+1-(ab)(ab)^*\}=ab(ab)^*(a+1-aa^*)(b+1-bb^*)$

$\Longrightarrow a^*b^*=aba^*b^*(a+1-aa^*)^{-1}(b+1-bb^*)^{-1}=(ab)(ab)^*\{ab+1-(ab)(ab)^*\}^{-1}=ab.$

Therefore U-regular ring with an involution $a\to a^*=au(a+1-au)^{-1}$ is $*$-U-regular ring, since involution is proper from (iv), because

$$a^*a=0\Longrightarrow aa^*a=0\Longrightarrow a=0 \text{ (from } R_4).$$

Hence we get the follwing theorem:

In a commutative U-regular ring R with $ab=0$, $aua=ava=a$ and $*:R\to R$ defined by $a\to a^*=au(a+1-au)^{-1}$, the following results are true.

(i) *R is $*$-U-regular ring.*

(ii) *every element it is partial isometry (from* R_4)

Thus Prop. 2 [1, pp. 10] *"In a*-ring with proper involution, b is a partial isometry*$\Longrightarrow$*b^*b is a projection" becomes trivial here.*

(iii) *A normal idempotent is projection.*

i.e. to prove $e=e^*$, *it suffices to prove* $e=e^*e$.

$(e^*e-e)^*(e^*e-e)=0 \Longrightarrow e^*e-e=0 \Longrightarrow e^*e=e$ (from iv).

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REFERENCE

[1] Sterling K-Berberian, *Bae *-ring*, Springer-Verlag, Band 195, (1971)