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ON MINIMAL BITOPOLOGICAL SPACES

By T.G. Raghavan and I.L. Reilly

1. Introduction

If X is a set and L is the set of all topologies on X, then it is well known that L is a complemented complete atomic lattice and L is non-modular if the cardinality of X is at least three, see Steiner [9] for example. In this paper we discuss the notion of minimal bitopological spaces in the partially ordered set $L \times L$, and obtain characterizations of such spaces for various bitopological properties. We now introduce the necessary terms and notation.

In the bitopological space $(X, \mathcal{T}_1, \mathcal{T}_2)$, a non-empty subset $\mathcal{F}$ of $\mathcal{T}_i$ $(i=1,2)$ is a $\mathcal{T}_i$ open filter if $\mathcal{F}$ satisfies

(i) $\phi \notin \mathcal{F}$

(ii) if $U, V \in \mathcal{F}$ then $U \cap V \in \mathcal{F}$

(iii) if $G \in \mathcal{T}_i$ and $G \supset U$ for some $U \in \mathcal{F}$ then $G \in \mathcal{F}$.

If, instead of (i), (ii) and (iii) $\mathcal{F}$ satisfies (i) and (iv) if $U, V \in \mathcal{F}$ there is a $W \in \mathcal{F}$ such that $W \subset U \cap V$, then $\mathcal{F}$ is a $\mathcal{T}_i$ open filterbase.

If B is a filterbase on $(X, \mathcal{T}_1, \mathcal{T}_2)$ the $\mathcal{T}_i$ adherence of B is defined by

$$\mathcal{T}_i \text{ ad}(B) = \cap \{\mathcal{T}_i \text{ cl } U : U \in B\}$$

(Throughout this paper the $\mathcal{T}_1$ closure of the set A is denoted by $\mathcal{T}_1$ cl A.)

The following definition was first given by Weston [10] who used the term "consistent". $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise Hausdorff if for each pair of distinct points x and y in X there is a $\mathcal{T}_1$ open set U and a $\mathcal{T}_2$ open set V such that $x \in U$, $y \in V$ and U and V are disjoint. If this disjointness condition can be replaced by the stronger condition $\mathcal{T}_2$ cl $U \cap \mathcal{T}_1$ cl $V = \phi$, then $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise Urysohn. $(X, \mathcal{T}_1, \mathcal{T}_2)$ is pairwise completely Hausdorff if for each pair of distinct points x and y in X there is a function $f : X \to [0,1]$ such that $f(x) \neq f(y)$ and f is $\mathcal{T}_i$ upper semicontinuous (henceforth abbreviated as u.s.c.) and

$\mathscr{T}_j$ lower semi-continuous (l. s. c.) where $i,j = 1,2$ and $i \neq j$.

The next two separation properties were introduced by Kelly [7]. $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise regular if for each point x in X and each $\mathscr{T}_i$ closed set P such that $x \notin P$ there is a $\mathscr{T}_i$ open set U and a $\mathscr{T}_j$ open set V disjoint from U such that $x \in U$ and $P \subset V$, where $i,j=1,2$ and $i \neq j$. $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise normal if for each $\mathscr{T}_1$ closed set A and $\mathscr{T}_2$ closed set B disjoint from A there is a $\mathscr{T}_1$ open set V containing B and a $\mathscr{T}_2$ open set U disjoint from V containing A. Following Fletcher [4] we say that $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise completely regular if for each $\mathscr{T}_i$ closed set C and each point $x \notin C$ there is a function $f: X \to [0,1]$ such that $f(C)=1$, $f(x)=0$ and f is $\mathscr{T}_i$ u. s. c. and $\mathscr{T}_j$ l. s. c., where $i,j=1,2$ and $i \neq j$.

2. Pairwise Hausdorff pairwise compact spaces

In this section we show that the delicate position of the compact Hausdorff topology in the lattice $\mathscr{L}$ of topologies on X carries over to the bitopological situation. Fletcher, Hoyle and Patty [5] call a cover $\mathscr{U}$ of $(X, \mathscr{T}_1, \mathscr{T}_2)$ pairwise open if $U \subset \mathscr{T}_1 \cup \mathscr{T}_2$ and if $\mathscr{U}$ contains a non-empty member of $\mathscr{T}_1$ and a non-empty member of $\mathscr{T}_2$. If every (countable) pairwise open cover of $(X, \mathscr{T}_1, \mathscr{T}_2)$ has a finite subcover the space is said to be pairwise (countably) compact. If $f: (X, \mathscr{T}_1, \mathscr{T}_2) \to (Y, \mathscr{S}_1, \mathscr{S}_2)$ is a map between bitopological spaces we say f is pairwise continuous (respectively; closed, open, homeomorphism) if $f: (X, \mathscr{T}_1) \to (Y, \mathscr{S}_1)$ and $f: (X, \mathscr{T}_2) \to (Y, \mathscr{S}_2)$ are continuous (respectively; closed, open, homeomorphisms).

We need the following two results. The first is due to Weston [10], and the second is Lemma 3 of Fletcher, Hoyle and Patty [5]. Note that in Proposition 2 A needs to be a proper subset of X.

PROPOSITION 1. *If $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise Hausdorff and A is a $\mathscr{T}_1$ compact subset of X then A is $\mathscr{T}_2$ closed.*

PROPOSITION 2. *If $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise compact and A is a $\mathscr{T}_1$ closed proper subset of X then A is $\mathscr{T}_2$ compact.*

THEOREM 1. *If $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise compact, $(Y, \mathscr{S}_1, \mathscr{S}_2)$ is pairwise Hausdorff and $f: (X, \mathscr{T}_1, \mathscr{T}_2) \to (Y, \mathscr{S}_1, \mathscr{S}_2)$ is a pairwise continuous surjection, then (a) f is pairwise closed (b) if f is injective, f is a pairwise homeomorphism.*

PROOF. (a) If A is a $\mathscr{T}_1$ closed proper subset of X, then it is $\mathscr{T}_2$ compact by Proposition 2. Hence $f(A)$ is $\mathscr{S}_2$ compact, and thus $\mathscr{S}_1$ closed by Proposition 1. If $A=X$, then $f(A)=Y$, and so is $\mathscr{S}_1$ closed. Thus $f:(X, \mathscr{T}_1)\to(Y, \mathscr{S}_1)$ is closed, and similarly for $f:(X, \mathscr{T}_2)\to(Y, \mathscr{S}_2)$.

(b) is immediate, since $f:(X,\mathscr{T}_i)\to(Y,\mathscr{S}_i)$ is a closed continuous bijection and therefore a homeomorphism, for $i=1,2$.

We call the bitopological space $(X,\mathscr{T}_1, \mathscr{T}_2)$ minimal pairwise Hausdorff if it is pairwise Hausdorff and if $(X,\mathscr{T}_3, \mathscr{T}_4)$ is pairwise Hausdorff with $\mathscr{T}_3\subset\mathscr{T}_1$ and $\mathscr{T}_4\subset\mathscr{T}_2$ then $\mathscr{T}_3=\mathscr{T}_1$ and $\mathscr{T}_4=\mathscr{T}_2$. Minimality of other bitopological properties is similarly defined, and maximality has the obvious definition.

THEOREM 2. *Every pairwise compact pairwise Hausdorff bitopological space is maximal pairwise compact and minimal pairwise Hausdorff.*

PROOF. Let $(X,\mathscr{T}_1, \mathscr{T}_2)$ be pairwise compact and pairwise Hausdorff. Then if $(X,\mathscr{S}_1,\mathscr{S}_2)$ properly contains $(X,\mathscr{T}_1, \mathscr{T}_2)$, $(X, \mathscr{S}_1, \mathscr{S}_2)$ is not pairwise compact. For otherwise, the previous theorem implies that the identity function $(X,\mathscr{S}_1, \mathscr{S}_2)\to(X, \mathscr{T}_1, \mathscr{T}_2)$ is a pairwise homeomorphism, contradicting the hypothesis of proper containment. Similarly, if $(X,\mathscr{T}_1, \mathscr{T}_2)$ properly contains $(X,\mathscr{U}_1, \mathscr{U}_2)$, then $(X,\mathscr{U}_1, \mathscr{U}_2)$ is not pairwise Hausdorff.

A similar set of results can be used to show that every pairwise countably compact pairwise Hausdorff first countable bitopological space is maximal pairwise countably compact and minimal pairwise Hausdorff first countable.

EXAMPLE. Let X be any infinite set, $\mathscr{T}_1$ be the cofinite topology on X and $\mathscr{T}_2$ be the discrete topology on X. Certainly $(X,\mathscr{T}_1, \mathscr{T}_2)$ is pairwise Hausdorff. We claim it is minimal pairwise Hausdorff. Since $(X,\mathscr{S}_1, \mathscr{S}_2)$ pairwise Hausdorff implies that $(X,\mathscr{S}_i)$ is T_1 for $i=1,2$ and $(X,\mathscr{T}_1)$ is minimal T_1, we cannot have $\mathscr{T}_3$ properly contained in $\mathscr{T}_1$ and still have $(X,\mathscr{T}_3, \mathscr{S}_2)$ pairwise Hausdorff, for $\mathscr{S}_2$ any topology on X. We fix $\mathscr{T}_1$, and consider a topology $\mathscr{T}_4$ properly contained in $\mathscr{T}_2$. Choose $\mathscr{T}_4$ to be one of the anti-atoms in the lattice $\mathscr{L}$ of topologies on X. So $\mathscr{T}_4$ is an ultraspace on X, so is either a principal or nonprincipal ultraspace, see Steiner [9]. An ultraspace is T_1 if and only if it is nonprincipal, and $(X,\mathscr{T}_1, \mathscr{T}_4)$ pairwise Hausdorff requires $(X, \mathscr{T}_4)$ to be $\mathscr{T}_1$. Thus $\mathscr{T}_4$ is a nonprincipal ultraspace on X, so that $\mathscr{T}_4$ is of the form $\mathscr{P}(X-\{x\})\cup\mathscr{U}$ where $\mathscr{P}(X-\{x\})$ is the collection of all subsets of X which do not contain x

and $\mathscr{U}$ is a nonprincipal ultrafilter on X. Now any member of $\mathscr{U}$ being an infinite subset of X meets every member of $\mathscr{T}_1$. Thus every $\mathscr{T}_1$ open set meets every T_4 open set that contains x. Thus $(X, \mathscr{T}_1, \mathscr{T}_4)$ is not pairwise Hausdorff, so that $(X, \mathscr{T}_1, \mathscr{T}_2)$ is minimal pairwise Hausdorff. We observe that$(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise compact, so that Theorem 2 immediately implies that it is minimal pairwise Hausdorff.

Fletcher, Hoyle and Patty [5, Example 4] give another example of a pairwise Hausdorff pairwise compact space.

3. Filterbase characterizations

In this section we obtain filterbase characterizations of several minimal bitopological spaces. Related topological results have been discussed extensively by Berri [1], Berri and Sorgenfrey [2], Bourbaki [3], Herrlich [6] and Scarborough [8].

In the following definitions, $\mathscr{B}$ is a $\mathscr{T}_i$ filterbase on $(X, \mathscr{T}_1, \mathscr{T}_2)$, and $i,j=1,2$, $i\neq j$.

DEFINITION 1. $\mathscr{B}$ is *$\mathscr{T}_i$ Urysohn with respect to $\mathscr{T}_j$* if for each point $p\notin \mathscr{T}_j$ ad $(\mathscr{B})$ there is a $\mathscr{T}_j$ open set U containing p and a set V in $\mathscr{B}$ such that $\mathscr{T}_i$ cl $U\cap \mathscr{T}_j$ cl $V=\phi$.

DEFINITION 2. $\mathscr{B}$ is *$\mathscr{T}_i$ completely Hausdorff with respect to $\mathscr{T}_j$* if for each point $p\notin \mathscr{T}_j$ ad $(\mathscr{B})$ there is a $\mathscr{T}_j$ open set U containing p, a set V in $\mathscr{B}$ and a function $f: X\to[0,1]$ such that $f(U)=1$, $f(V)=0$ and f is $\mathscr{T}_j$ l.s.c. and $\mathscr{T}_i$ u.s.c.

DEFINITION 3. $\mathscr{B}$ is *$\mathscr{T}_i$ regular with respect to $\mathscr{T}_j$* if for each $U\in\mathscr{B}$ there is a $V\in\mathscr{B}$ such that $\mathscr{T}_j$ cl $V\subset U$.

DEFINITION 4. $\mathscr{B}$ is *$\mathscr{T}_i$ completely regular with respect to $\mathscr{T}_j$* if for each $U\in\mathscr{B}$ there is a $V\in\mathscr{B}$ and a function $f: X\to[0,1]$ such that $f(V)=0$, $F(X-U)=1$ and f is $\mathscr{T}_i$ u.s.c. and $\mathscr{T}_j$ l.s.c.

THEOREM 3. *If $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise Hausdorff, the following are equivalent.*

(a) $(X, \mathscr{T}_1, \mathscr{T}_2)$ is minimal pairwise Hausdorff.

(b) Every $\mathscr{T}_i$ open filterbase on X with a unique $\mathscr{T}_j$ adherent point p is $\mathscr{T}_i$ convergent to p, where $i,j=1,2$, $i\neq j$.

REMARK. Condition (b) of Theorem 3 states that if $\mathscr{B}_1$ is a $\mathscr{T}_1$ open filterbase with a unique $\mathscr{T}_2$ adherent point p, and if $\mathscr{B}_2$ is a $\mathscr{T}_2$ open filterbase with the same point p as its unique $\mathscr{T}_1$ adherent point then $\mathscr{B}_1$ is $\mathscr{T}_1$ convergent to p and $\mathscr{B}_2$ is $\mathscr{T}_2$ convergent to p. We observe that this sharing of the point p applies also to our later characterizations of minimal bitopological spaces in Theorems 4, 5 and 6.

PROOF of Theorem 3.

(a) implies (b). Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be minimal pairwise Hausdorff and $\mathscr{B}_i$ be a $\mathscr{T}_i$ open filterbase with unique $\mathscr{T}_j$ adherent point p. Let $\mathscr{T}_j\langle x\rangle$ denote the collection of all $\mathscr{T}_j$ open neighbourhoods of x. Let $\mathscr{S}_i=\{U : U\in\mathscr{T}_i, p\notin U\}\cup\{V\cup B : V\in \mathscr{T}_i\langle p\rangle, B\in\mathscr{F}_i\}$ where $\mathscr{F}_j$ is the filter generated by $\mathscr{B}_i$, for $i=1,2$. Clearly $(X, \mathscr{S}_1, \mathscr{S}_2)$ satisfies $\mathscr{S}_1\subset\mathscr{T}_1$ and $\mathscr{S}_2\subset\mathscr{T}_2$. Let x be a point of X distinct from p. Then x is not a $\mathscr{T}_j$ adherent point of $\mathscr{F}_i$, so that there is a $V\in\mathscr{T}_j\langle x\rangle$ such that $V\cap B=\phi$ for some $B\in\mathscr{F}_i$. Since $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise Hausdorff, there is a $U\in\mathscr{T}_i\langle p\rangle$ and a $W\in\mathscr{T}_j\langle x\rangle$ such that $U\cap W=\phi$. Now $V\cap W\in\mathscr{S}_j\langle x\rangle$ since $V\cap W\in\mathscr{T}_j\langle x\rangle$ and $p\notin V\cap W$. Moreover, $U\cup B\in\mathscr{S}_i\langle p\rangle$, and $(V\cap W)\cap(U\cup B)=\phi$. Also if x and y are different points of X each distinct from p it is clear that there is a $\mathscr{S}_i$ open set U and a $\mathscr{S}_j$ open set V with $x\in U$, $y\in V$ and $U\cap V=\phi$. Hence $(X, \mathscr{S}_1, \mathscr{S}_2)$ is pairwise Hausdorff. By the minimality of $(X, \mathscr{T}_1, \mathscr{T}_2)$ we have $\mathscr{S}_1=\mathscr{T}_1$ and $\mathscr{S}_2=\mathscr{T}_2$. In particular, $\mathscr{T}_i\langle p\rangle=\mathscr{S}_i\langle p\rangle$, so that $U\in\mathscr{T}_i\langle p\rangle$ implies $U\in\mathscr{S}_i\langle p\rangle$. Hence $U\supset B$ for some $B\in\mathscr{F}_i$, so that the filterbase $\mathscr{B}_i$ is $\mathscr{T}_i$ convergent to p.

(b) implies (a). Let $(X, \mathscr{S}_1, \mathscr{S}_2)$ be pairwise Hausdorff such that $\mathscr{S}_1\subset\mathscr{T}_1$ and $\mathscr{S}_2\subset\mathscr{T}_2$. We show that $\mathscr{S}_i=\mathscr{T}_i$ by proving that $\mathscr{S}_i\langle p\rangle=\mathscr{S}_i\langle p\rangle$ for each point p in $X, i=1,2$. Let p be any point in X, then $\mathscr{S}_i\langle p\rangle$ is a $\mathscr{T}_i$ open filterbase. Now p is the only $\mathscr{T}_j$ adherent point of $\mathscr{S}_i\langle p\rangle$, since $\mathscr{T}_j$ cl $A\subset\mathscr{S}_j$ cl A and $(X, \mathscr{S}_1, \mathscr{S}_1)$ is pairwise Hausdorff. Therefore, by condition (b), $\mathscr{S}_i\langle p\rangle$ is $\mathscr{T}_i$ convergent to p. Hence $\mathscr{S}_i\langle p\rangle\supset\mathscr{T}_i\langle p\rangle$. Clearly $\mathscr{S}_i\langle p\rangle\subset\mathscr{T}_i\langle p\rangle$, so that $\mathscr{S}_i\langle p\rangle=\mathscr{T}_i\langle p\rangle$ as desired.

In the following theorem, G denotes one of the properties Urysohn, completely Hausdorff, regular or completely regular.

THEOREM 4. *Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be a pairwise G bitopological space. Then the following are equivalent.*

(a) $(X, \mathscr{T}_1, \mathscr{T}_2)$ is minimal pairwise G.

(*b*) *Every* $\mathcal{T}_i$ *open filterbase on* X *which is* $\mathcal{T}_i$ *G with respect to* $\mathcal{T}_j$ *and which has a unique* $\mathcal{T}_j$ *adherent point* p, *is* $\mathcal{T}_i$ *convergent to* p, *where* $i,j=1,2$, $i\neq j$.

PROOF. (a) implies (b). Let $(X,\mathcal{T}_1,\mathcal{T}_2)$ be minimal pairwise G, and $\mathcal{B}_i$ be a $\mathcal{T}_i$ open filterbase on X which is $\mathcal{T}_i$ G with respect to $\mathcal{T}_j$ and which has a unique $\mathcal{T}_j$ adherent point p. Let $\mathcal{F}_i$ be the filter generated by B_i, and as in the proof of Theorem 3 we define the topology $\mathcal{S}_i$ on X by $\mathcal{S}_i=\{U: U\in\mathcal{T}_i,\ p\notin U\}\cup\{V\cup B: V\in\mathcal{T}_i\langle p\rangle,\ B\in\mathcal{F}_i\}$, for $i=1,2$. If we show that $(X,\mathcal{S}_1,\mathcal{S}_2)$ is pairwise G, then $\mathcal{S}_i=\mathcal{T}_i$ by the minimality of $(X,\mathcal{T}_1,\mathcal{T}_2)$. Again as in the proof of Theorem 3, this implies that $\mathcal{B}_i$ is $\mathcal{T}_i$ convergent to p. We discuss the cases separately.

(i) $G=$Urysohn. It is clear that if x and y are distinct points of X both different from p then they can be separated by appropriate sets. Now let x be distinct from p. Since $\mathcal{B}_i$ is $\mathcal{T}_i$ Urysohn with respect to $\mathcal{T}_j$ there is a $U\in\mathcal{T}_j\langle x\rangle$ such that $\mathcal{T}_i$ cl $U\cap\mathcal{T}_j$ cl $B=\phi$ for some B in $\mathcal{B}_i$. Since $(X,\mathcal{T}_1,\mathcal{T}_2)$ is pairwise Urysohn there is a $V\in\mathcal{T}_i\langle p\rangle$ and a $W\in\mathcal{T}_j\langle x\rangle$ such that $\mathcal{T}_i$ cl $W\cap\mathcal{T}_j$ cl $V=\phi$. Then $\mathcal{T}_i$ cl $(U\cap W)=\mathcal{S}_i$ cl$(U\cap W)$, and since $p\in V\cup B$ we have $\mathcal{T}_j$ cl $(V\cup B)=\mathcal{S}_j$ cl $(V\cup B)$. Furthermore, $\mathcal{S}_i$ cl $(U\cap W)\cap\mathcal{S}_j$ cl $(V\cup B)=\phi$. Hence $(X,\mathcal{S}_1,\mathcal{S}_2)$ is pairwise Urysohn.

(ii) $G=$Completely Hausdorff. First suppose that x is distinct from p. Since $\mathcal{B}_i$ is $\mathcal{T}_i$ completely Hausdorff with respect to $\mathcal{T}_j$ there is a $U\in\mathcal{T}_j\langle x\rangle$, some $B\in\mathcal{B}_i$ and a function $f: X\to[0,1]$ such that $f(U)=1$, $f(B)=0$ and f is $\mathcal{T}_j$ l.s.c. and $\mathcal{T}_i$ u.s.c. Then $f(p)=0$. For otherwise, let $f(p)=k>0$. Let $L=[k/2,1]$. Since f is $\mathcal{T}_j$ l.s.c., $f^{-1}(L)$ is $\mathcal{T}_j$ open, and $f^{-1}(L)\cap B=\phi$ implies that $p\notin\mathcal{T}_j$ cl B, which contradicts $\{p\}=\mathcal{T}_j$ ad $(\mathcal{B}_i)$. Now f is $\mathcal{S}_j$ l.s.c. at the point p, for $p\in X$ and $f(X)\subset[0,1]$. Since f is $\mathcal{T}_i$ u.s.c., for each $k>0$ there is a $\mathcal{T}_i$ open set V such that $p\in V$ and $f(V)\subset[0,k)$. Now $f(B)=0$ implies $f(V\cup B)\subset[0,k)$, and $V\cup B$ is $\mathcal{S}_i$ open, so that f is $\mathcal{S}_i$ u.s.c. at the point p. If z is some point other than p, two cases arise. If $f(z)=0$, then $z\in X$, $f(X)\subset[0,1]$ and X is $\mathcal{S}_j$ open. If $f(z)=k>0$, then since f is $\mathcal{T}_j$ l.s.c. there is a $\mathcal{T}_j$ open set V such that $z\in V$ and $f(V)\subset(r,1]$, for each r such that $0<r<k$. Moreover $p\notin V$ as $f(p)=0$. Thus V is $\mathcal{S}_j$ open. Hence, in both cases, f is $\mathcal{S}_j$ l.s.c. at z. Similarly we can show that f is $\mathcal{S}_i$ u.s.c. at z.

Now suppose that x and y are both different from p. Since $\mathcal{B}_i$ is $\mathcal{T}_i$ completely Hausdorff with respect to $\mathcal{T}_j$, there is a $U\in\mathcal{T}_j\langle x\rangle$, some $B\in\mathcal{B}_i$ and a function $f: X \to [0,1]$ such that $f(U)=1$, $f(B)=0$ and f is $\mathscr{T}_i$ u.s.c. and $\mathscr{T}_j$ l.s.c. If $f(y) \neq 1$ then f will serve as the desired function. If $f(y)=1$ the situation is more difficult. Since $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise completely Hausdorff there is a function $g: X \to [0,1]$ which is $\mathscr{T}_i$ u.s.c. and $\mathscr{T}_j$ l.s.c. and $g(x) \neq g(y)$. Now let $h(z)=f(z) \cdot g(z)$ for all $z \in X$. Then h is $\mathscr{T}_i$ u.s.c. and $\mathscr{T}_j$ l.s.c., and $h(x)=g(x)$ while $h(y)=g(y)$ so that $h(x) \neq h(y)$. As before $f(p)=0$, so $h(p)=0$. Then h is $\mathscr{S}_j$ l.s.c. at p, since $h(X) \subset [0,1]$ and $X \in \mathscr{S}_j$. Since h is $\mathscr{T}_i$ u.s.c., for each $k>0$ there is a $\mathscr{T}_i$ open set V with $p \in V$ and $h(V) \subset [0,k)$. Moreover, $h(B)=0$ since $f(B)=0$. Thus $h(V \cup B) \subset [0,k)$, and $V \cup B$ is $\mathscr{S}_i$ open, so that h is $\mathscr{S}_i$ u.s.c. at p. If z is some point other than p, a similar argument to that above shows that h is $\mathscr{S}_j$ l.s.c. at z and $\mathscr{S}_i$ u.s.c. at z.

Thus any pair of points of X can be separated by a suitable function, so that $(X, \mathscr{S}_1, \mathscr{S}_2)$ is pairwise completely Hausdorff.

(iii) G=regular. The pairwise regularity of $(X, \mathscr{S}_1, \mathscr{S}_2)$ at any point other than p follows immediately from that of $(X, \mathscr{T}_1, \mathscr{T}_2)$. Let $V \cup B$ be a $\mathscr{S}_i$ open neighbourhood of p, where $V \in \mathscr{T}_i \langle p \rangle$ and $B \in \mathscr{B}_i$. Since $(X, \mathscr{S}_1, \mathscr{S}_2)$ is pairwise regular there is a $Q \in \mathscr{T}_i \langle p \rangle$ such that $\mathscr{T}_j$ cl $Q \subset V$. Since $\mathscr{B}_i$ is $\mathscr{T}_j$ regular with respect to $\mathscr{T}_i$ there is a $W \in \mathscr{B}_i$ such that $\mathscr{T}_j$ cl $W \subset B$. Now $p \in Q$ so that $\mathscr{T}_j$ cl $Q = \mathscr{S}_j$ cl Q, and $p \in \mathscr{T}_j$ cl W so that $\mathscr{T}_j$ cl $W = \mathscr{S}_j$ cl W. Thus $Q \cup W \in \mathscr{S}_i \langle p \rangle$ and $\mathscr{S}_j$ cl $(Q \cup W) \subset V \cup B$. Thus $(X, \mathscr{S}_1, \mathscr{S}_2)$ is pairwise regular at p.

(iv) G=completely regular. First suppose that A is $\mathscr{S}_i$ closed and $p \notin A$. Let $X-A=V \cup B$ where $V \in \mathscr{S}_i \langle p \rangle$ and $B \in \mathscr{B}_i$. Now $\mathscr{B}_i$ is $\mathscr{T}_j$ completely regular with respect to $\mathscr{T}_i$ so there is a $B_1 \in \mathscr{B}_i$ and a function $f: X \to [0,1]$ such that $f(B_1)=0$, $f(X-B)=1$ and f is $\mathscr{T}_i$ u.s.c and $\mathscr{T}_j$ l.s.c. Since $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise completely regular there is a function $g: X \to [0,1]$ such that $g(p)=0$, $g(X-V)=1$ and g is $\mathscr{T}_i$ u.s.c. and $\mathscr{T}_j$ l.s.c. We define $h: X \to [0,1]$ by $h(z)=f(z) \cdot g(z)$, for $z \in X$. Clearly, h is $\mathscr{S}_i$ u.s.c. and $\mathscr{S}_j$ l.s.c. at any point z distinct from p, $h(p)=0$ and $h(A)=1$. Since $h(p)=0$ and $h(X) \subset [0,1]$, h is $\mathscr{S}_j$ l.s.c. at p. Now for each ε such that $0<\varepsilon<1$ since f is $\mathscr{T}_i$ u.s.c. there is a $W \in \mathscr{T}_i \langle p \rangle$ with $f(W) \subset [0, \varepsilon)$. Then $f(W \cup B_1) \subset [0, \varepsilon)$. Hence $h(W \cup B_1) \subset [0, \varepsilon)$ and $W \cup B_1 \in \mathscr{S}_i$, so that h is $\mathscr{S}_i$ u.s.c. at p.

Now suppose that x is a point of X distinct from p, that A is $\mathscr{S}_j$ closed and $x \notin A$. Let $U_1=X-A$ so that $U_1 \in \mathscr{S}_i \langle x \rangle$. Since x is not a $\mathscr{T}_j$ adherent point of $\mathscr{B}_i$ there is a $V_1 \in \mathscr{T}_j \langle x \rangle$ such that $V_1 \cap B = \phi$ for some $B \in \mathscr{B}_i$. Let $W=U_1$

$\cap V_1$. Since $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise completely regular there is a function $f' : X \to [0,1]$ such that $f'(x)=1$, $f'(X-W)=0$ and f' is $\mathscr{T}_i$ u.s.c. and $\mathscr{T}_j$ l.s.c. Now $\mathscr{B}_i$ is $\mathscr{T}_i$ completely regular with respect to $\mathscr{T}_j$ so there is a $B' \in \mathscr{B}_i$ and a function $g' : X \to [0,1]$ such that $g'(B')=0$, $g'(X-B)=1$ and g' is $\mathscr{T}_i$ u.s.c. and $\mathscr{T}_j$ l.s.c. We define $k : X \to [0,1]$ by $k(z)=f'(z) \cdot g'(z)$ for $z \in X$. Then $k(x)=1$ and $k(X-W)=0$ so that $k(A)=0$. We now prove that k is $\mathscr{S}_j$ u.s.c. and $\mathscr{S}_j$ l.s.c. If z is any point distinct from p, two cases arise. If $k(z)=0$, then $k(X) \subset [0,1]$ so that k is $\mathscr{S}_j$ l.s.c. at z. For any $r>0$, since k is $\mathscr{T}_i$ u.s.c. there is a $V_2 \in \mathscr{T}_i \langle z \rangle$ such that $k(V_2) \subset [0, r)$. Furthermore, we choose V_2 so that $p \notin V_2$. Then $V_2 \in \mathscr{S}_i \langle z \rangle$, and k is $\mathscr{S}_i$ u.s.c. at z. If $k(z) \neq 0$, let r be such that $0<r<k(z)$. Let r_1 and r_2 be such that $f'(z) \in (r_1, 1]$, $g'(z) \in (r_2, 1]$ and $r_1 \cdot r_2 = r$. Now $g'(p)=0$, otherwise $p \notin \mathscr{T}_j$ cl B'. Since f' is $\mathscr{T}_j$ l.s.c. there is a $V_3 \in \mathscr{T}_j \langle z \rangle$ such that $f'(V_3) \subset (r_1, 1]$. Since g' is $\mathscr{T}_j$ l.s.c. there is a $V_4 \in \mathscr{T}_j \langle z \rangle$ such that $g'(V_4) \subset (r_2, 1]$. Then $k[V_3 \cap V_4] \subset (r, 1]$, $z \in V_3 \cap V_4$ and $p \in V_3 \cap V_4$, so that $V_3 \cap V_4 \in \mathscr{S}_j \langle z \rangle$ and k is $\mathscr{S}_j$ l.s.c. at z. Similarly we can show that k is $\mathscr{T}_i$ u.s.c. at z. Now $k(p)=0$, so it is clear that k is $\mathscr{S}_j$ l.s.c. at p. For each $r>0$ since g' is $\mathscr{T}_i$ u.s.c. there is a $H \in \mathscr{T}_i \langle p \rangle$ such that $g'(H) \subset [0, r)$. Then $g'(H \cup B') \subset [0, r)$, so that $k(H \cup B') \subset [0, r)$ and hence k is $\mathscr{S}_i$ u.s.c. at p.

(b) implies (a). Suppose that $(X, \mathscr{S}_1, \mathscr{S}_2)$ is pairwise G and that $\mathscr{S}_i \subset \mathscr{T}_i$ for $i=1,2$. Let p be any point of X. Consider the $\mathscr{T}_i$ open filterbase $\mathscr{S}_i \langle p \rangle$. Certainly p is a $\mathscr{T}_j$ adherent point of $\mathscr{S}_i \langle p \rangle$, and it is unique since $(X, \mathscr{S}_1, \mathscr{S}_2)$ is pairwise G. We now show that $\mathscr{S}_i \langle p \rangle$ is $\mathscr{T}_i$ G with respect to $\mathscr{T}_j$. Consider the case G=Urysohn. (For the other three properties, the proof follows similar lines, and hence is omitted.) Take any point $x \notin \mathscr{T}_j$ ad $(\mathscr{S}_i \langle p \rangle)$, so $x \neq p$. Since $(X, \mathscr{S}_1, \mathscr{S}_2)$ is pairwise Urysohn there is a $U \in \mathscr{S}_i \langle p \rangle$ and a $V \in \mathscr{S}_j \langle x \rangle \subset \mathscr{T}_j \langle x \rangle$ such that $\mathscr{S}_j$ cl $U \cap \mathscr{S}_i$ cl $V = \phi$. But $\mathscr{T}_j$ cl $U \subset \mathscr{S}_j$ cl U and $\mathscr{T}_i$ cl $V \subset \mathscr{S}_i$ cl V, so that $\mathscr{T}_j$ cl $U \cap \mathscr{T}$ cl $V=\phi$, and hence $\mathscr{S}_i$ $\langle p \rangle$ is $\mathscr{T}_i$ Urysohn with respect to $\mathscr{T}_j$. Thus (b) implies that $\mathscr{S}_i \langle p \rangle$ is $\mathscr{T}_i$ convergent to p. Hence $\mathscr{S}_i$ $\langle p \rangle \supset \mathscr{T}_i \langle p \rangle$, so that $\mathscr{S}_i \langle p \rangle = \mathscr{T}_i \langle p \rangle$, for each point p in X, and $i=1,2$. Thus $\mathscr{S}_i = \mathscr{T}_i$ for $i=1,2$, and $(X, \mathscr{T}_1, \mathscr{T}_2)$ is minimal pairwise G.

Using similar arguments we can obtain filterbase characterizations of minimal first countable bitopological spaces. A filterbase on X is countable if the filter it generates is a countable collection of subsets of X.

THEOREM 5. *Let $(X, \mathscr{T}_1, \mathscr{T}_2)$ be a pairwise Hausdorff bitopological space. Then the following are equivalent.*

(a) *$(X, \mathscr{T}_1, \mathscr{T}_2)$ is minimal pairwise Hausdorff first countable.*

(b) *Every countable $\mathscr{T}_i$ open filterbase on X with a unique $\mathscr{T}_j$ adherent point p, is $\mathscr{T}_i$ convergent to p, where $i, j, = 1, 2, i \neq j$.*

THEOREM 6. *If $(X, \mathscr{T}_1, \mathscr{T}_2)$ is pairwise G first countable the following are equivalent.*

(a) *$(X_i, \mathscr{T}_1, \mathscr{T}_2)$ is minimal pairwise G first countable.*

(b) *Every countable T_i open filterbase on X which is $\mathscr{T}_i$ G with respect to $\mathscr{T}_j$ and which has a unique $\mathscr{T}_j$ adherent point p, is $\mathscr{T}_i$ convergent to p, where $i, j = 1, 2, i \neq j$.*

Madurai University
Madurai
India

and

University of Auckland
Auckland
New Zealand

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