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ON A TENSOR FIELD f OF TYPE (1,1) SATISFYING $f^k \pm f^r = 0$, $(k \geq 2r)$

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1. $f^k \pm f^r = 0$ structures

Consider M^n to be an n-dimensional differentiable manifold of class C^∞ and let there be given a tensor field $f \neq 0$ of type (1,1) and of class C^∞ satisfying

$$f^k \pm f^r = 0, \quad (k \geq 2r), \tag{1.1}$$

such that

$$(2 \text{ rank } f - \text{rank } f^{k-r}) = \dim M^n.$$

Let us define the operators l and m by

$$l \overset{\text{def}}{=\!=} \mp f^{k-r}, \quad m \overset{\text{def}}{=\!=} I \pm f^{k-r}, \tag{1.2}$$

I denoting the identity operator, then we have:

THEOREM 1.1. *For a tensor field $f \neq 0$, satisfying* (1.1), *the operators l, m defined by* (1.2) *and applied to the tangent space at a point of the manifold are complementary projection operators.*

PROOF. We have

$$l^2 = (\mp f^{k-r})^2 = f^{2k-2r} = f^k \cdot f^{k-2r} = (\mp f^r) f^{k-2r} = \mp f^{k-r} = l,$$

and similarly we can show that $m^2 = m$, $lm = ml = 0$ and $l + m = I$. Thus the theorem is proved.

Let L and M be the complementary distributions corresponding to the projection operators l and m respectively and let the rank of f be equal to p (a constant), then $\dim L = (2p - n)$ and $\dim M = (2n - 2p)$, $(n \leq 2p \leq 2n)$.

A structure with the above properties is called an $f(k, \pm r)$-structure of rank p and the manifold M^n with this structure is called an $f(k, \pm r)$-manifold.

THEOREM 1.2. *For a tensor field f satisfying* (1.1) *and the operators l, m defined by* (1.2), *l acts on f^r as an identity operator and m acts on both f^r and $f^{(k-r)/2}$ as a null operator. Also $f^{(k-r)/2}$ acts on L either as an almost complex structure operator or as an almost product structure operator, according as* *we take $f(k,r)$ or $(k,-r)$ structure.*

PROOF. It can be easily proved that $f^r\, l=f^r$, $f^r\, m=0$, $f^{(k-r)/2}m=0$, $f^{(k-r)}\, l=\mp l$, which is the contention of our theorem.

THEOREM 1.3. *If $F=f^{(k-r)/2}$, then $F(k,\pm r)$-structure of maximal rank is an almost complex structure (almost product structure), respectively.*

PROOF. If the rank of F is maximal $p=n$, therefore dim $L=n$ and dim $M=0$. Thus $m=0$, which implies theorem 1.3.

THEOREM 1.4. *If $F=f^{(k-r)/2}$, then F $(k,+r)$-structure of minimal rank is an almost tangent structure.*

PROOF. If the rank of F is minimal, $2p=n$, therefore dim $L=0$ and dim $M=n$, Thus $l=0$, which shows that $f^{(k-r)}=0$. Hence the theorem is proved.

THEOREM 1.5. *For a tensor field f satisfying f (k,r)-structure $(m-f^{(k-r)/2})(m+f^{(k-r)/2})=I$ and satisfying $f(k,-r)$-structure $(l-f^{(k-r)/2})(l+f^{(k-r)/2})=0$.*

PROOF. We can prove this theorem by simple calculation.

THEOREM 1.6. *If in M^n there is given a tensor field $f\neq 0$, $f^{(k-r)}\neq I$ of class C^∞ satisfying $f(k,-r)$-structure, then M^n admits an almost product structure $\eta=2f^{(k-r)}-I$.*

PROOF. Since $\eta=2f^{(k-r)}-I$, therefore we can easily prove that $\eta^2=I$, which proves the theorem.

THEOREM 1.7. *Let p and q be tensors defined by*

$$p\overset{\text{def}}{=\!=}(m-f^{k-r}),\quad q\overset{\text{def}}{=\!=}(m+f^{k-r}),$$

then

i) *for an $f(k,r)$-structure we have*
$p^2=q^2=I$, $pq=qp$, $p^3\pm q^3=p\pm q$, $pl=l$, $p^2l=l$, $pm=m$, $p^2m=m$, $ql=-l$, $q^2l=l$, $qm=m$, $q^2m=m$, $pql=-l$, $pqm=m$.

ii) *for an $f(k,-r)$-structure we have*
$p^2=q^2=q$, $pq=p$, $p^3\pm q^3=p\pm q$, $pl=-f^{(k-r)}$, $p^2l=l$, $pm=m$, $p^2m=m$, $ql=l$, $q^2l=l$, $qm=m$, $q^2m=m$.

PROOF. These results can be proved by simple calculation.

2. Metric for $f(k, \pm r)$-structures

THEOREM 2.1. *If in an n-dimensional manifold M^n, there is given a tensor field $f \neq 0$ of rank p and satisfies above structures, then there exist complementary distributions L of dimension $(2p-n)$ and M of dimension $(2n-2p)$ and a positive definite Riemannian metric g with respect to which L and M are orthogonal such that*

$$h_j^t \ h_i^s \ g_{ts} + m_{ji} = g_{ji}, \quad \text{where } f^{(k-r)/2} \stackrel{\text{def}}{=\!=} h.$$

Also we have

i) *For $f(k, r)$-structure $h_{ji} = -h_{ij}$ and the rank p of f is even,*

ii) *For $f(k, -r)$-structure $h_{ji} = h_{ij}$, and the rank p of f is odd.*

PROOF. Let f_i^h , l_i^h , m_i^h be the local components of the tensors f, l, m respectively. Let u_a^h $(a, b, c, \cdots = 1, 2, \cdots, 2p-n)$ be $2p-n$ mutually orthogonal unit vectors in L and u_A^h $(A, B, C, \cdots = 2p-n+1, \cdots, n)$ be $2n-2p$ mutually orthogonal unit vectors in M, then we have

$$l_i^h \ u_b^i = u_b^h, \quad l_i^h \ u_B^i = 0, \tag{2.1}$$

$$m_i^h \ u_b^i = 0, \quad m_i^h \ u_B^i = u_B^h,$$

Since we know that $f^{(k-r)/2} m = 0$, therefore we find

$$h_i^l \ u_B^i = 0. \tag{2.2}$$

Let $(v_i^a, \ v_i^A)$ be the matrix inverse of $(u_b^h, \ u_B^h)$, then v_i^a and v_i^A are both components of linearly independent covariant vectors which satisfy

$$v_i^a \ u_b^i = \delta_b^a, \quad v_i^a \ u_B^i = 0, \tag{2.3)a}$$

$$v_i^A \ u_b^i = 0, \quad v_i^A \ u_B^i = \delta_B^A, \tag{2.3)b}$$

$$v_i^a \ u_a^h + v_i^A \ u_A^h = \delta_i^h. \tag{2.3)c}$$

Using (2.3) in (2.1) we easily obtain

$$l_i^h \ v_h^a = v_i^a, \quad l_i^h \ v_h^A = 0, \tag{2.4}$$

$$m_i^h \ v_h^a = 0, \quad m_i^h \ v_h^A = v_i^A,$$

which yields

$$h_i^l \ v_l^A = 0, \tag{2.5)a}$$

$$l_i^h = v_i^a \ u_a^h, \tag{b}$$

c) $m_i^h = v_i^A \ u_A^h.$

Now following Yano [2] we have a globally defined positive definite Riemannian metric with respect to which $(u_b^h,\ u_B^h)$ form an orthogonal frame such that

$$v_j^a = a_{ji}\ u_a^i,\quad v_j^A = a_{ji}\ u_A^i,$$

where

(2.6) $$a_{ji} = v_j^a\ v_i^a + v_j^A\ v_i^A.$$

By Putting $l_{ji} = l_j^t\ a_{ti},\ m_{ji} = m_j^t\ a_{ti}$, we easily get

(2.7) $$l_{ji} + m_{ji} = a_{ji}$$

Also we can easily verify the following relations:

$$l_j^t\ l_i^s\ a_{ts} = l_{ji},\quad l_j^t\ m_i^s a_{ts} = 0,\quad m_j^t\ m_i^s a_{ts} = m_{ji}$$

By Putting

(2.8) $$g_{ji} = \frac{1}{2}(a_{ji} + h_j^t\ h_i^s\ a_{ts} + m_{ji}),$$

we have a globally defined positive definite Riemannian metric which satisfies

(2.9) $$v_j^A = g_{ji}\ u_A^i,\quad m_{ji} = m_j^t\ g_{ti}.$$

From (2.9) we can see that the distributions L and M which are orthogonal with respect to a_{ji} are still orthogonal with respect to g_{ji} and u_A^h, which are mutually orthogonal unit vectors with respect to a_{ji} are also mutually orthogonal with respect to g_{ji}. Thus it is easy to verify that the tensor g_{ji} satisfies

(2.10) $$h_j^t\ h_i^s\ g_{ts} + m_{ji} = g_{ji},$$

which proves first part of the theorem.

i) Since for an $f(k,r)$-structure we have $-h^2 = I - m$, therefore we can write

(2.11) $$-h_j^t\ h_t^i + m_j^i = \delta_j^i.$$

Now putting $h_{it} = h_i^s\ g_{st}$, we get from (2.10) and (2.11) the following equations

(2.12) $$h_j^t\ h_{it} + m_{ji} = g_{ji}$$

and

(2.13) $$-h_j^t\ h_{ti} + m_{ji} = g_{ji}.$$

Substracting (2.13) from (2.12), we get

(2.14) $$h_j^t\ (h_{it} + h_{ti}) = 0.$$

Since $h_j^t \neq 0$, equation (2.14) shows that h_{it} is a skew-symmetric tensor of rank p and p must be even.

ii) For $f(k, -r)$-structure we have $h^2 = I - m$, which similarly implies

$$h_j^t (h_{it} - h_{ti}) = 0, \tag{2.15}$$

showing that h_{it} is symmetric tensor of rank p and p must be odd.

3. Some properties of $f(k, r)$-structure

THEOREM 3.1. *If L is integrable, then the subspace v^A=constant for a $f(k, r)$-structure admits an almost complex structure.*

PROOF. If ξ^i are local coordinates in the original manifold then the distribution L is defined locally by

$$m_i^h d\xi^i = 0 \quad \text{or } v_i^A d\xi^i = 0. \tag{3.1}$$

The integrability condition of (3.1) can be given by

$$l_j^t \, l_i^s (\partial_t m_s^h - \partial_s m_t^h) = 0, \tag{3.2}$$

where $\partial_t = \partial / \partial \xi^t$,

Let the distribution L be integrable then denoting by $v^A(\xi)$=constant, the equations of integral manifolds we can choose v_i^A in such a way that

$$v_i^A = \partial_i \, v^A. \tag{3.3}$$

If η^a are the parameters and the parametric equations of one of the integral manifolds are $\xi^h = \xi^h(\eta^a)$, then we have

$$B_b^h \, v_h^A = 0, \tag{3.4}$$

where

$$B_b^h = \partial_b' \xi^h \quad (\partial_b = \partial / \partial \eta^b).$$

Thus we can choose u_A^h in such a way that the matrix inverse to $(B_b^h, \ u_A^h)$ is (B_i^a, v_i^A) such that we have

$$B_i^a \, B_b^i = \delta_b^a, \quad B_i^a \, u_B^i = 0, \tag{3.5}$$

$$v_i^A \, B_b^i = 0, \quad v_i^A \, u_B^i = \delta_B^A$$

and

$$l_i^h = B_i^a \, B_a^h, \quad m_i^h = v_i^A \, u_A^h. \tag{3.6}$$

If we put

$$(3.7) \qquad 'h_b^a = B_b^i B_l^a h_i^l,$$

we can easily verify that

$$(3.8) \qquad 'h_b^a \, 'h_c^b = -\delta_c^a,$$

which proves theorem 3.1.

Let ∇_j and $'\nabla_c$ be the covariant derivatives in the enveloping space and the subspace respectively, then the Nijenhuis tensor for the almost complex structure $'h_b^a$ is

$$(3.9) \qquad 'N_{cb}^a = 'h_c^d \, '\nabla_d \, 'h_b^a - 'h_b^d \, '\nabla_d \, 'h_c^a - ('\nabla_c \, 'h_b^d - '\nabla_b \, 'h_c^d) 'h_d^a.$$

Substituting (3.7) in (3.9) we get

$$(3.10) \qquad 'N_{cb}^a = B_c^j B_b^i B_h^a N_{ji}^h,$$

where

$$(3.11) \qquad N_{ji}^b = h_j^l \nabla_l h_i^p - h_i^l \nabla_l h_j^p - (\nabla_j h_i^l - \nabla_i h_j^l) h_l^p.$$

DEFINITION 3.1. When the distribution L is integrable and the almost complex structure induced on the integral manifold is also integrable, we say that the $f(k,r)$-structure is *partially integrable.*

THEOREM 3.2. *A necessary and sufficient condition for an $f(k,r)$-structure to be partially integrable is that the Nijenhuis tensor satisfies:*

$$N_{pq}^h l_j^p l_i^q = 0.$$

PROOF. When $f(k,r)$-structure is partially integrable we have

$$(3.12) \qquad B_c^j B_b^i B_h^a N_{ji}^h = 0.$$

From (3.11) we have

$$(3.13) \qquad N_{ji}^l m_l^p = -h_j^k h_i^l (\nabla_k m_l^p - \nabla_l m_k^p),$$

which in case of the distribution L being integrable yields

$$(3.14) \qquad N_{ji}^l m_l^p = 0.$$

If we contract equation (3.12) with $B_j^c B_i^b B_a^h$ we get

$$(3.15) \qquad N_{pq}^h l_j^p l_i^q = 0.$$

Conversely suppose that $f(k,r)$-structure satisfies (3.15), then from (3.12) we have

$$l_j^t l_i^s (\nabla_t h_s^l - \nabla_s h_t^l) h_l^p = 0,$$

which is equivalent to

$$l_j^t \, l_i^s (\nabla_t m_s^h - \nabla_s m_t^h) = 0. \tag{3.16}$$

Thus the distribution L is integrable and we can induce an almost complex structure $'h_b^a$ on the integral manifold. For the Nijenhuis tensor of this almost complex structure we have

$$'N_{cb}^a = 0, \tag{3.17}$$

which proves the theorem.

THEOREM 3.3. *A necessary and sufficient condition for an n-dimensional manifold* M^n *to admit a tensor field* $f \neq 0$ *of type* (1,1) *and of rank p such that* $f^{2q+r} + f^r = 0$, *(r odd) is that p be even and the group of tangent bundle of the manifold be reduced to group* $S(2s=2tq) \times 0(n-2s)$.

PROOF. Let

$$u_{s+1}^t = h_i^t \, u_1^i, \quad u_{s+2}^t = h_i^t \, u_2^i, \cdots, u_{2s}^t = h_i^t \, u_s^i, \tag{3.18}$$

be $2s$ mutually orthogonal unit vectors in L then with respect to the orthogonal frame (u_b^t, u_B^t) the tensors g_{ji} and h_{ji} have components

$$g = \begin{pmatrix} E_s & 0 & 0 \\ 0 & E_s & 0 \\ 0 & 0 & E_{n-2s} \end{pmatrix}, \quad h = f^{\left(\frac{k-r}{2}\right)} = \begin{pmatrix} 0 & E_s & 0 \\ -E_s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.19}$$

where E_s denotes the $s \times s$ unit matrix.

Let f be a structure (f, k) such that $p=2s$ and $k=2q+r$ then following Kim [1] it is observed that $f^r u_1 \neq u_1$ and s is divisible by q. Let $s=tq$. If we put $f^r u_i = u_{i+t}$ and $f^r \, u_{i+2s-rt} = -u_i$, for $i=1,2,\cdots, s$ then $hu_i = f^q u_i = u_{i+tq} = u_{i+s}$ and $h^2 u_i = f^{2q} u_i = f^r u_{i+(2q-r)t} = f^r u_{i+2s-rt} = -u_i$.

Thus we can write

$$f^r = \begin{pmatrix} 0 & E_{2s-rt} & 0 \\ -E_{rt} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.20}$$

Now we take another adapted frame $(\bar{u}_b^h, \bar{u}_B^h)$ with respect to which the metric tensor g_{ji} and h_{ji} have the same components as (3.19) and put $\bar{u}_b^h = \gamma_b^a u_a^h$, $\bar{u}_B^h = \gamma_B^A u_A^h$, then following Yano [2], the orthogonal matrix

$$F=(\gamma_b^a)=\begin{pmatrix} S & 0 \\ 0 & 0_{n-2s} \end{pmatrix},$$

where

$$S=\begin{pmatrix} S_{11} & S_{12} & \cdots & S_{1q} \\ S_{21} & S_{22} & \cdots & S_{2q} \\ \vdots & \vdots & \cdots & \vdots \\ S_{q1} & S_{q2} & \cdots & S_{qq} \end{pmatrix}$$

and S_{ij} is a $t\times t$ matrix, takes the form

(3.21) $$F=\begin{pmatrix} \bar{S} & 0 \\ 0 & 0_{n-2s} \end{pmatrix},$$

where

$$\bar{S}=\begin{pmatrix} S_{11} & S_{12} & \cdots & S_{1q} \\ -S_{1q} & S_{11} & \cdots & S_{1q-1} \\ \vdots & \vdots & \cdots & \vdots \\ -S_{12} & -S_{13} & \cdots & S_{11} \end{pmatrix}$$

Let S be the tangent group defined by $\bar{S}$ in (3.21), then the group of tangent bundle of the manifold can be reduced to $S\times O(n-2s)$, then we can define a positive definite Riemannian metric g and tensors f and $h=f^q$ of type (1, 1) and of rank $2s$ as tensors having (3.19) and (3.20) as components with respect to adapted frames. Then we have

$$f^q=\begin{pmatrix} 0 & E_s & 0 \\ -E_s & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad f^{2q}=\begin{pmatrix} -E_s & 0 & 0 \\ 0 & -E_s & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and $f^{2q+r}+f^r=0$, which proves theorem 3.3.

REMARKS 1. Similar results can be established for the structure $f(k, -r)$ also.

2. Integrability conditions and some other properties of these structures are being studied in a subsequent paper.

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