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ON MONO-SOURCES AND INITIAL SOURCES IN THE CATEGORY OF τ-ALGEBRAS

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1. Introduction

The aim of this note is to find out the relation between the mono-sources and the initial sources in universal algebra. Here we have studied to investigate whether an initial source is a mono-source in universal algebra and under which condition the proposition holds. And in theorem 3.1 and 3.2, we will prove that initial source does not, in general, implies mono-source in τ-**Alg** but it does in **Grp.**

2. Preliminaries

This section is a collection of basic definitions and results which will be needed in the ensuing sections, and we omit the proofs of them, which will be found in [2, Chap. 0, 1].

DEFINITION 2.1. A *source* in a category **C** is a pair $(X, (f_i)_I)$, where X is a C-object, I is a class, and $(f_i\colon X \longrightarrow X_i)_I$ is a family of **C**-morphisms each with domain X.

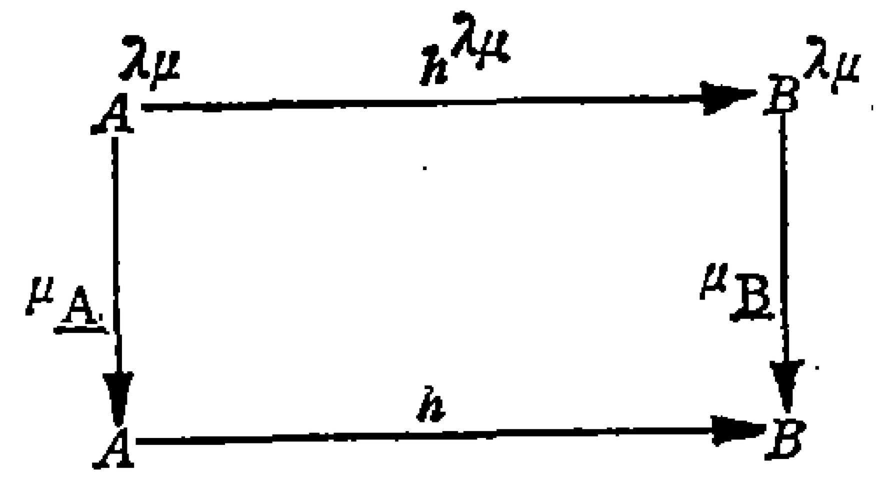
DEFINITION 2.2. In **Set**, a source $(X, (f_i)_I)$ is said to *separate points of* X if whenever x and y are distinct points in X, there is an index $j \in I$ such that $f_j(x) \neq f_j(y)$.

DEFINITION 2.3. (1) A map $f\colon X^\lambda \longrightarrow X$, where X is any set and λ any ordinal number, is called an *operation* on X. In particular, λ is called the *arity* of f, and f is called a λ*-ary operation* on X.

(2) A *type* τ of universal algebras is a family $\tau = (\lambda_\mu)_{\mu \in M}$ of ordinals, indexed by a set M, Given a type $\tau = (\lambda_\mu)_{\mu \in M}$, a *universal algebra of type* τ is a pair (X, φ), where X is a set and $\varphi = (\mu_X)_{\mu \in M}$ a family of operations on X, each μ_X of arity λ_μ.

In the following, the algebera will be denoted by **X** if there are no confusion about φ, and X will be called the *underlying set of* $(X, \varphi)=\mathbf{X}$.

DEFINITION 2.4. A *homomorphism* from **A** to **B** is a map h: $A \longrightarrow B$ such that the following diagram commutes for each $\mu \in M$.



DEFINITION 2.5. A source $(X, (f_i)_I)$ is called a *mono-source* provided that the f_i can be simultaneously cancelled from the left; i.e., provided that for any pair $Y \overset{r}{\underset{s}{\rightrightarrows}} X$ of morphisms such that $f_i r = f_i s$ for each i, it follows that $r=s$.

DEFINITION 2.6. Let **A** and **B** be categories and U: $\mathbf{A} \longrightarrow \mathbf{B}$ a functor. A source $(A, (f_i\colon A \longrightarrow A_i)_I)$ in **A** is called U-initial if for any source $(A', (g_i\colon A' \longrightarrow A_i)_I)$ in **A** and any **B**-morphism h: $UA' \longrightarrow UA$ with $(Uf_i)h = Ug_i$ for each $i \in I$, there exists a unique **A**-morphism $\bar{h}: A' \longrightarrow A$ with $U\bar{h}=h$ and $f_i \bar{h} = g_i$ for each $i \in I$.

PROPOSITION 2.7. *Let* τ-**Alg** *be the category of* τ-*algebras, and* F: τ-**Alg** $\longrightarrow$ **Set** *a forgetful functor. Then in* τ-**Alg**, *every mono-source is* F-*initial.*

For the proof of prop. 2.7 refer to [2, chap. 2, 2.9]

The above property is one of the essential properties of algebraic structure, with which the algebraic structure is distinguished from other structures, for instance, the topological structure.

PROPOSITION 2.8. *Every mono-source is point separating.*

PROOF. Given a family of homomorphisms h_i: $\mathbf{A} \longrightarrow \mathbf{B}_i$ such that, for each i, $h_i(a)=h_i(b)$ for some distinct $a, c \in A$. Consider the subalgebra **C** generated by the element (a, c). (the subalgebra of the product algebra $\mathbf{A} \times \mathbf{A}$) Let u, v: $\mathbf{C} \longrightarrow \mathbf{A}$ be the homomorphisms such that $u=p_1 j$ and $v=p_2 j$, where j is an inclusion homomorphism of **C** into $\mathbf{A} \times \mathbf{A}$. Then $h_i u(a, c) = h_i(a) = h_i(c) = h_i v(a, c)$, and hence $h_i u = h_i v$. But $u \neq v$, and this implies that (h_i) is not a mono-source. This completes the proof.

The proof given here is a slight modification of that given in [2].

REMARK. As we can see in the proof of above proposition, if a subcategory $\mathscr{L}$ of the category of all τ-algebras admits finite product and subalgebra, i.e., finitely productive and hereditary, the point-separating source in $\mathscr{L}$ is equivalent to the mono-source in $\mathscr{L}$.

3. Main results and remarks

THEOREM 3.1. *In the category of τ-algebras, an initial source is not necessarily a mono-source.*

PROOF. We will prove the theorem by showing a counter example. Let $X=\{x, y, z\}$ and $Y=\{u, v\}$, and let the operations on X and Y be the constant maps to z and v resp. Let f: $X \longrightarrow Y$ be defined by $f(x)=f(y)=u$, $f(z)=v$, then f is a homomorphism and f is not one-to-one. Let's prove f is initial. Let **T** be any algebra of the same type and g: $\mathbf{T} \longrightarrow \mathbf{X}$ any homomorphism. If fg is a homomorphism, then $\mu_Y f^{\lambda_\mu} g^{\lambda_\mu} = fg\mu_T$. But since the map of the left side is a constant map to v, $g\mu_T(t_i)=z$ for each $(t_i)\in T^{\lambda_\mu}$, and this implies $\mu_X g^{\lambda_\mu} = g\mu_T$. This completes the proof.

THEOREM 3.2. *A source $(G, (h_i))$ in the category of groups is initial iff it is a mono-source.*

PROOF. ($\Leftarrow$) Trivial by 2.7.
($\Rightarrow$) Suppose not. Then there exist two distinct elements x, y in X such that $h_i(x)=h_i(y)$ for each i. Let a map g: $\{e\} \longrightarrow G$ be defined by $g(e)=xy^{-1}$. Then gh_i is a homomorphism but g is not, which contradicts the fact that (h_i) is initial

In the above two theorems, initial means F-initial.

REMARK. Using the fact that every mono-source is initial, we have been able to show that most of basic constructions of algebras are strongly related with those sets, namely underlying sets of subalgebras and product algebras are in fact subsets and product sets of their underlying sets respectively, and that basic properties of homomorphisms are also related with those of (underlying) maps.

It is left unsolved whether initiality implies monosource in other algebraic structures, say, semigroups, monoids, etc.

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