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CHARACTERIZATIONS OF NEARLY COMPACT SPACES

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1. Introduction

In this paper nearly compact spaces are characterized using a type of convergence for filterbases and nets. Additionally, Hausdorff nearly compact spaces are characterized using the technique of Prof. Kasahara [5]. A space is said to be nearly compact if and only if each regular-open cover admits a finite subcover [7, Theorem 2.1]. Throughout this paper, $\overline{A}$ will denote the closure of A and A^0 will denote the interior.

2. Preliminaries

DEFINITION 1. A filterbase $\mathscr{F}=\{A_a\}$ is said to *w-converge to a point* $x\in X$ if and only if for every regular-open set V containing x there exists an $A_{a(x)}\in\mathscr{F}$ such that $A_{a(x)}\subset V$.

DEFINITION 2. A filterbase $\mathscr{F}=\{A_a\}$ is said to *w-accumulate to a point* $x\in X$ if and only if for every regular-open set V containing x, $A_a\cap V\neq\phi$ for every $A_a\in\mathscr{F}$.

DEFINITION 3. A net $\{x_a\}$ is said to *w-converge (accumulate) to a point* $x\in X$ if and only if $\{x_a\}$ is eventually (frequently) in every regular-open set containg x.

It is apparent upon inspection that filterbases and nets are "equivalent" in the sense of w-convergence (accumulation). The theorems immediately following are easily proven and are stated without proof.

THEOREM 1. *If a filterbase (net) w-converges to a point $x\in X$, then it w-accumulates to $x\in X$.*

THEOREM 2. *Let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two filterbases in X with $\mathscr{F}_2$ subordinate to $\mathscr{F}_1$. Then if $\mathscr{F}_2$ w-accumulates to a point $x\in X$, $\mathscr{F}_1$ w-accumulates to $x\in X$.*

THEOREM 3. *A maximal filterbase $\mathscr{F}$ w-accumulates to a point $x\in X$ if and only if $\mathscr{F}$ w-converges to $x\in X$.*

DEFINITION 4. A function $f:X\to Y$ is *almost continuous* if and only if the inverse image of every regular-open set of Y is open in X. (An equivalent definition is obtained if one replaces regular-open and open with regular-closed and closed. See [8, Theorem 22].)

THEOREM 4. *A function $f: X \to Y$ is almost continuous at $x \in X$ if and only if for every net $\{x_a\}$ in X such that $\{x_a\}$ converges to x in the usual sense, the net $\{f(x_a)\}$ w-converges to $f(x)$.*

3. Characterizations of arbitrary nearly compact spaces

Using the concept of w-convergence, the following characterizations are offered.

THEOREM 5. *For a topological space, the following are equivalent:*

(1) *X is nearly compact.*

(2) *If $\{F_a\}$ is a family of regular-closed sets such that $\bigcap F_a = \phi$, then there exists a finite subfamily such that $\bigcap_{i=1}^{n} F_{a_i} = \phi$.*

(3) *Every filterbase has a w-accumulation point.*

(4) *Every maximal filterbase w-converges to some point $x \in X$.*

PROOF. (1⇒4). Let $\mathscr{F} = \{A_a\}$ be a maximal filterbase such that $\mathscr{F}$ does not w-converge to any point; hence, $\mathscr{F}$ does not w-accumulate to any point. Then for every $x \in X$, there exists a regular open set G_x containing x and $A_{a(x)} \in \mathscr{F}$ such that $A_{a(x)} \cap G_x = \phi$. Therefore, $\{G_x\}$ covers X and by hypothesis, there exists a finite subcover $\{G_{x_i}\}$. Since $\mathscr{F}$ is a filterbase, there exists $A_0 \subset \bigcap_{i=1}^{n} A_{a(x_i)}$, $A_0 \neq \phi$, $A_0 \in \mathscr{F}$. But

$$A_0 = A_0 \cap X = A_0 \cap \left(\bigcup_{i=1}^{n} G_{x_i}\right) = \bigcup_{i=1}^{n} (A_0 \cap G_{x_i}) \subset \bigcup_{i=1}^{n} (A_{a(x_i)} \cap G_{x_i}) = \phi,$$

contradicting the fact that $A_0 \neq \phi$. Therefore, $\mathscr{F}$ must have a w-convergent point, and hence must converge by Theorem 3.

(4⇒3). Every filterbase is contained in a maximal filterbase.

(3⇒2). Let $\{F_a\}$ be a family of regular closed sets such that $\bigcap F_a = \phi$. Suppose for each finite subfamily, $\bigcap_{i=1}^{n} F_{a_i} \neq \phi$. Then $\mathscr{F} = \{$all finite intersections of elements of $\{F_a\}\}$ forms a filterbase. By hypothesis, $\mathscr{F}$ w-accumulates to some point $x_0 \in X$. Since $\bigcap F_a = \phi$, there exists an $a(x)$ such that $x_0 \notin F_{a(x)}$. Therefore, $x_0 \in X - F_{a(x)}$, a regular-open set. But since $\mathscr{F}$ w-accumulates to x_0 and $F_{a(x)}$

$\in \mathscr{F}$, we must have $F_{a(x)} \cap (X - F_{a(x)}) \neq \phi$, an impossibility. Therefore, there must be a finite subfamily satisfying the finite intersection property.

(2⇒1). Let $\{V_a\}$ be a regular-open cover of X. Then $\cap (X - V_a) = \phi$. By assumption, there exists a finite subfamily such that $\bigcap_{i=1}^{n} (X - V_{a_i}) = \phi$. Therefore, $X = \bigcup_{i=1}^{n} V_{a_i}$, and X is nearly compact.

THEOREM 6. *For a topological space, the following are equivalent:*

(1) *X is nearly compact.*

(2) *Every net has a w-accumulation point.*

(3) *Every ultra-net has a w-convergent point.*

4. Characterizing Hausdorff nearly compact spaces

We now turn our attention to Hausdorff nearly compact spaces.

DEFINITION 5. The graph of a function $f: X \to Y$, denoted $G(f)$, is said to be *r-closed* if and only if for every $(x, y) \in X \times Y$ such $f(x) \neq y$, there exists regular-open sets U and V containing x and y, respectively, such that $f(U) \cap V = \phi$.

THEOREM 7. *Let $f: X \to Y$ be any function between any two topological spaces. If $G(f)$ is r-closed, then $\{(x_a, f(x_a))\}$ w-converging to (x, y) in $X \times Y$ implies that $f(x) = y$.*

PROOF. To the contrary, suppose $f(x) \neq y$. Then by hypothesis, there exists regular-open sets U and V containing x and y, respectively, such that $f(U) \cap V = \phi$. Since $\{(x_a, f(x_a))\}$ must eventually be in $U \times V$, we know that $\{x_a\}$ is eventually in U and $\{f(x_a)\}$ is eventually in V. But $\{f(x_a)\}$ must also eventually be in U, an impossibility since $f(U) \cap V = \phi$. Therefore, $f(x) = y$.

It is well known that a function $f: X \to Y$ with a closed graph and Y compact is necessarily continuous. Following is the analogue to this result.

THEOREM 8. *Let $f: X \to Y$ be a function with an r-closed graph. If Y is nearly compact, then f is almost continuous.*

PROOF. Let $K \subset Y$ be a regular-closed set. Let $x \in \overline{f^{-1}(K)}$. Then there exists a net $\{x_a\} \subset f^{-1}(K)$ such that $\{x_a\}$ converges to x. Since $\{f(x_a)\} \subset K$, $\{f(x_a)\}$ w-accumulates to some point $y \in Y$ by near compactness (Theorem 6). Therefore, $\{(x_a, f(x_a))\}$ w-accumulates to the point $(x, y) \in X \times Y$. Since there exists a subnet of $\{(x_a, f(x_a))\}$ that w-converges to (x, y), we have by Theorem 7 that $f(x)=y$. If $y \notin K$, then $y \in Y-K$, a regular-open set. But $\{f(x_a)\} \subset K$, hence cannot be frequently in $Y-K$. Therefore, $y \in K$, $x \in f^{-1}(K)$, and f is almost continuous.

We now modify the technique of Prof. Kasahara in order to characterize Hausdorff nearly compact spaces. Let $\mathscr{S}$ denote the class of spaces containing the class of Hausdorff completely normal and fully normal spaces.

THEOREM 9. *A Hausdorff space Y is nearly compact if and only if for every space $X \in \mathscr{S}$, each function $f: X \to Y$ with an r-closed graph is almost continuous.*

PROOF. In view of Theorem 8, only necessity remains to be proven. Suppose Y is not nearly compact. Then by Theorem 6, there exists a net $\{y_a\}$ in Y with no w-accumulation point. Let D be the directed set associated with $\{y_a\}$ and form the set $X = D \cup \{\infty\}$, $\infty \notin D$. Define $T_a = \cup \{b \in D \mid b \geq a\}$. We now topologize X by declaring the open sets to be any subset of D and sets of the form $T_a \cup \{\infty\}$; i.e., $\mathscr{P}(D) \cup \{T_a \cup \{\infty\}\}$. Let $y_0 \in Y$ be an arbitrary point and define f as follows:

$$\begin{aligned} f(x) &= y_a, \text{ if } x = a \in D \\ &= y_0, \text{ if } x = \infty. \end{aligned}$$

We proceed to show that f has an r-closed graph but is not almost continuous. Let $(x, y) \in X \times Y$ such that $f(x) \neq y$. First, suppose $x \neq \infty$. By the Hausdorff property, there exists a regular-open set V such that $y \in V$, $f(x) \notin V$. Then, since $\{x\}$ is a regular-open set, we have $f(\{x\}) \cap V = \phi$. Now assume $x = \infty$. Since the net $\{y_a\}$ does not w-accumulate to $y \in Y$, this implies the existence of a regular open set V containing y and not containing $f(x)$ such that $\{y_a\}$ is eventually outside of V; i.e., there exists a $b \in D$ such that $T_b \cap V = \phi$. Since $T_b \cup \{\infty\}$ is a regular open set, we have $f(T_b \cup \{\infty\}) \cap V = \phi$. Hence, $G(f)$ is r-closed.

The identity map $i: D \to D \subset X$ is a net in X converging to ∞ in the usual sense. But it is now quite obvious that the image of this net, namely $\{y_a\}$, does not w-converge to y_0. Therefore, by Theorem 4, f is not almost continuous, and the theorem follows from contraposition.

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REFERENCES

[1] Bourbaki, N., *General topology*, Part I, Addison-Wesley Publishing Company, 1966.

[2] Carnahan, D., *On locally nearly-compact spaces*, Boll. Un. Mat. Ital., (4), 6 (1972), pp. 147—153.

[3] Dugundji, J., *Topology*, Allyn and Bacon, Boston 1966.

[4] Herrington, L. L., *Some properties preserved by the almost continuous function*, Boll. Un. Mat. Ital., (4), 10 (1974), pp. 556—568.

[5] Kashara, S., *Characterizations of compactness and countable compactness*, Proc. Japan Acad., Vol. 49, no. 7 (1973) pp. 523—524.

[6] Long, P. E., & Carnahan, D. A., *Comparing almost-continuous functions*, Proc. Amer. Math. Soc., 38 (1973), pp. 413—418.

[7] Singal, M. K., & Mathur, A., *On nearly-compact spaces*, Boll. Un. Mat. Ital. (4), 2 (1969), pp. 702—710.

[8] Singal, M. K., & Singal, A. R., *On almost-continuous mappings*, Yokohama Math. J., 16 (1968), pp. 63—73.