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NOTE ON HANKEL TRANSFORMS

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Titchmarsh first defined the Hankel transform $\mathfrak{H}_\mu$ for $\mu \geq -\frac{1}{2}$ by

$$\Phi(y)=(\mathfrak{H}_\mu \varphi(x))(y)=\int_0^\infty \varphi(x)\sqrt{xy}\, J_\mu\,(xy)\,dx \tag{1}$$

where $J_\mu(x)$ is the Bessel function of the first kind and proved the following inversion formula ([7 : pp. 240—242]):

THEOREM 1. *If $\varphi \in L^1(0,\infty)$ is of bounded variation in a neighborhood of the point x, then for $\mu \geq -\frac{1}{2}$*

$$\frac{1}{2}\{\varphi(x+0)+\varphi(x-0)\}=(\mathfrak{H}_\mu^{-1}\Phi(y))(x)=\int_0^\infty \Phi(y)\sqrt{xy}\, J_\mu(xy)\,dy \tag{2}$$

It was extended to distributions by Zemanian as follows ([10—12]). Let H_μ be the space of smooth functions defined on $(0,\infty)$ satisfying the inequalities

$$\gamma^\mu_{p,q}(\varphi)=\sup_{0<x<\infty}\left|x^p(x^{-1}D)^q(x^{-\mu-1/2}\varphi(x))\right|<\infty,\quad p,q=0,1,2,\cdots\cdots$$

equipped with the topology generated by the seminorms $\{\gamma^\mu_{p,q}\}^\infty_{p,q=0}$. Then the Hankel transform $\mathfrak{H}_\mu$ defined by (1) is an automorphism on H_μ. If the generalized Hankel transform $\mathfrak{H}_\mu'$ is defined by

$$\langle \mathfrak{H}_\mu' f,\ \varphi\rangle=\langle f, \mathfrak{H}_\mu\varphi\rangle \tag{3}$$

where f belongs to the dual space H_μ' and $\varphi \in H_\mu$, then $\mathfrak{H}_\mu'$ is an automorphism on the dual space H_μ'. Define the operator $N_\mu=x^{\mu+1/2}D_x x^{-(\mu+1/2)}$ with the inverse N_μ^{-1} given by

$$N_\mu^{-1}\varphi(x)=x^{\mu+1/2}\int_\infty^x y^{-(\mu+1/2)}\varphi(y)dy$$

Let m be a positive integer greater than $-\mu-1/2$ for any given real number μ. Then the Hankel transform of arbitrary order $\mathfrak{H}_{\mu,m}$ is defined by

$$\Phi(y)=(\mathfrak{H}_{\mu,m}\varphi(x))(y)=(-1)^m y^{-m}\mathfrak{H}_{\mu+m}N_{\mu+m-1}\cdots N_\mu\,\varphi(x) \tag{4}$$

with its inverse Hankel transform $\mathfrak{H}_{\mu,m}^{-1}$ given by

$$\varphi(x)=(\mathfrak{H}_{\mu,m}^{-1}\Phi(y))(x)=(-1)^m N_\mu^{-1}\cdots N_{\mu+m-1}^{-1}\mathfrak{H}_{\mu+m}y^m\ \Phi(y)$$

Replacing $\mathfrak{H}_\mu$ in the right hand side of (3) by $\mathfrak{H}_{\mu,m}$ we ob[illegible] [11 : pp. 764—765])

THEOREM 2. *For any real number μ, the Hankel transform $\mathfrak{H}_{\mu,m}$ defined by* (4) *is an automorphism on H_μ, and hence the generalized Hankel transform $\mathfrak{H}_\mu'$ defined by* (3) *is an automorphism on the dual space H_μ'.*

Motivated by Hirschman, Jr and Haimo's work on variation diminishing Hankel transforms ([2], [3]), Schwartz later on defined his Hankel transform $\mathscr{H}_\mu$ for $\mu\geq-\frac{1}{2}$ by ([6 : p. 713])

$$\Psi(y)=(\mathscr{H}_\mu\phi(x))(y)=\int_0^\infty \phi(x)\mathscr{J}_\mu(xy)\ dm(x) \tag{5}$$

where $dm(x)=[2^\mu\Gamma(\mu+1)]^{-1}x^{2\mu+1}\ dx$ and $\mathscr{J}_\mu(x)=2^\mu\Gamma(\mu+1)x^{-\mu}J_\mu(x)$. Let $L(0,\infty)$ be the space of $L^1(0,\infty)$-integrable functions with respect to the Radon measure $dm(x)$. He then proved the following inversion formula ([6 : pp. 713—715]):

THEOREM 3. *Let ϕ belong to $L(0,\infty)$ and let*

$$\int_0^1 \phi(x)\ x^{\mu+1/2}\ dx<\infty.$$

If ϕ is of bounded variation in a neighborhood of the point x, then

$$\frac{1}{2}\{\phi(x+0)+\phi(x-0)\}=(\mathscr{H}_\mu^{-1}\ \Psi(y))(x)=\int_0^\infty \Psi(y)\ \ \mathscr{J}_\mu(xy)\ dm(y) \tag{6}$$

In [5 : p. 432] we raised the question on relations between the two Hankel transforms (1), (5) and their respective inversion formulas (2), (6). In this paper we prove that they are essentially the same. A straightforward computation reveals that (5) and (6) are reduced respectively to

$$\Psi(y)=(\mathscr{H}_\mu\phi(x))(y)=\int_0^\infty \phi(x)\sqrt{xy}\ (x/y)^{\mu+1/2}\ J_\mu(xy)\ dy \tag{7}$$

$$\frac{1}{2}\{\phi(x-0)+\phi(x+0)\}=(\mathscr{H}_\mu^{-1}\ \Psi(y))(x)=\int_0^\infty \Psi(y)\sqrt{xy}\ (y/x)^{\mu+1/2}J_\mu(xy)dy \tag{8}$$

To give a refined form of Theorem 3, we need the following definition.

DEFINITION. For any real number $p \geq 0$, the space $E_p(\Omega)$ consists of L^1-integrable functions defined on an open subset $\Omega \subset (0, \infty)$ with respect to the Radon measure $x^p\, dx$ where dx is the Lebesgue measure.

From the definition we have $E_0(\Omega) = L^1(\Omega)$, $E_{2\mu+1}(\Omega) = L(\Omega)$ and in particular $x^{-(\mu+1/2)} \cdot E_{\mu+1/2}(0, \infty) = L^1(0, \infty)$. We shall call a function φ in $L^1(0, \infty)$ an $E_{\mu+1/2}$-bounded variation if $x^{\mu+1/2}\, \varphi$ is of bounded variation in $L^1(0, \infty)$. Then Theorem 3 is refined as follows:

THEOREM 3′. *Let $\phi \in E_{\mu+1/2}(0, \infty)$ be an $E_{\mu+1/2}$-bounded variation in a neighborhood of the point x, then* (7) *and* (8) *are inverse to each other under the Hankel transform* (5).

Now we prove the main thoerem. Hereafter μ is any real number $\geq -\frac{1}{2}$.

THEOREM 4. (a) *Theorem* 1 *implies Theorem* 3′ *under the mapping $\varphi \to (x/y)^{-(\mu+1/2)}\, \varphi$. In other words*

$$(\mathfrak{H}_\mu \varphi(x))(y) = (\mathcal{H}_\mu (x/y)^{-(\mu+1/2)}\, \varphi(x))(y) \tag{9}$$

and

$$(\mathfrak{H}_\mu^{-1}\, \Phi(y))(x) = (\mathcal{H}_\mu^{-1}\, (y/x)^{-(\mu+1/2)}\, \Phi(y))(x) \tag{10}$$

(b) *Theorem* 3′ *implies Theorem* 1 *under the mapping $\phi \to (x/y)^{\mu+1/2}\phi$. In other words,*

$$(\mathcal{H}_\mu \phi\,(x))\,(y) = \mathfrak{H}_\mu((x/y)^{\mu+1/2}\, \phi(x))(y)$$

and

$$(\mathcal{H}_\mu^{-1} \Psi\,(y))\,(x) = \mathfrak{H}_\mu((y/x)^{\mu+1/2} \Psi(y))(x)$$

PROOF. Since the proof of (a) and (b) are identical we prove (a) only. Let φ satisfy the assumptions of Theorem 1 and consider the mapping $\varphi \to (x/y)^{-(\mu+1/2)}\varphi$. Since

$$\|\varphi\|_{L^1} = y^{-(\mu+1/2)} \|(x/y)^{-(\mu+1/2)} \varphi\|_{E_{\mu+1/2}}$$

this mapping is injective from $L^1(0, \infty)$ into $E_{\mu+1/2}(0, \infty)$. Moreover φ is of bounded variation in $L^1(0, \infty)$ if and only if $(x/y)^{-(\mu+1/2)}\, \varphi$ is of $E_{\mu+1/2}$-bounded variation in $E_{\mu+1/2}(0, \infty)$. Thus $(x/y)^{-(\mu+1/2)}\, \varphi$ satisfies the assumptions of Theorem 3′. This completes the proof.

Since the mapping $\varphi \to (x/y)^{-(\mu+1/2)}\varphi$ is an isomorphism from H_μ onto $x^{\mu+1/2}H_\mu$, an application of $\mathfrak{H}_\mu$ and $\mathfrak{H}_\mu'$ on the space H_μ and on its dual space H_μ' respectively allows us to extend Theorem 4 to distributions. Thus we have

THEOREM 5. (a) *For* $\mu \geq -\frac{1}{2}$, *the two Hankel transforms* $\mathfrak{H}_\mu$ *and* $\mathscr{H}_\mu$ *are equivalent on the spaces* H_μ *and* $x^{\mu+1/2}H_\mu$ *respectively, that is for all* $\varphi \in H_\mu$

$$(\mathfrak{H}_\mu\varphi(x))(y) = \mathscr{H}_\mu((x/y)^{-(\mu+1/2)}\varphi(x))(y)$$

(b) *For* $\mu \geq -\frac{1}{2}$, *the two generalized Hankel transforms* $\mathfrak{H}_\mu$ *and* $\mathscr{H}_\mu'$ *are equivalent on the dual spaces* H_μ' *and* $(x^{\mu+1/2}H_\mu)'$ *respectively in a sense of* (3), *namely for each* $f \in H_\mu'$ *and for all* $\varphi \in H_\mu$

$$\langle \mathfrak{H}_\mu' f, \varphi \rangle = \langle \mathscr{H}_\mu'(x/y)^{-(\mu+1/2)}f, \varphi \rangle$$

Theorem 5 answers our previous questions ([5 : pp. 431—432]).

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