Kyungpook Math. J.
Volume 17, Number 1
June, 1977

ON A CONJECTURE OF SAEKS ON MINIMUM EXTENSION RESOLUTION SPACE

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Let H be a Hilbert space, and let E^t be a resolution of identity where t ranges over a LCA topological group G. Following [1]—[4], a family of shift operators is defined as a class of unitary operators such that

$$U^t(B^A) \subset B^{A+t}$$

for all $t \in G$ and for all Borel subsets A where B^A is the range of $E^A = \int_A dE(\lambda)$, and a group of shift operators as a class of shift operators such that

$$U^{t-s} = [U^t]\,[U^s]^{-1}$$

for all t and s in G. The resolution space (H, E^t, G) is called a uniform resolution space if it admits a group of shift operators. Let μ be a σ-finite positive Borel measure on G. Then Saeks proved ([1 : p. 324]) that (H, E^t, G) admits a group of shift operators if and only if μ is equivalent to a Haar measure m on G. From now on we shall exclusively deal with uniform resolution space.

Now let M be a class of positive finite Borel measures on G and let u be a spectral multiplicity function defined on M. The following theorem was proved by Saeks ([1 : pp. 324—328]).

THEOREM 1. *Every uniform resolution space (H, E^t, G) is equivalent to $L_2(m, H)^c$ where m is a Haar measure on G and c is a cardinal number.*

In fact if $\{\mu^k\}$ is a canonical representation for the spectral multiplicity function u ([7 : pp. 76—87, 106—108]), $\{\mu^k\}$ actually consists of a single Borel measure and so $c = u(\mu^k)$. It follows that if $(\underline{H}, \underline{E}^t, G)$ is a minimal extension uniform resolution space in a sense that $\underline{H}$ is spanned by the family of elements $\{U^n x(t): x(t) \in H,\ t \in G,\ n \in Z\}$ where U is a unitary extension of a contraction T on H ([10]—[11]), then

$$(H, E^t, G) \cong L_2(m, H)^{c_1} \qquad (1)$$

$$(\underline{H}, \underline{E}^t, G) \cong L_2(\underline{m}, \underline{H})^{c_2} \qquad (2)$$

where m, $\underline{m}$ are Haar measures on G and c_1, c_2 are cardinal numbers. In [1 :

p. 334] Saeks conjectured that $c_1 \geq c_2$. However the inequality turned out to be equality, namely

THEOREM 2. *Let* (H, E^t, G) *be a uniform resolution space and let* $(\underline{H}, \underline{E}^t, G)$ *be its minimal extension uniform resolution space. Then*

$$c_1 = c_2$$

where c_1 *and* c_2 *are given by the equations* (1) *and* (2) *respectively.*

PROOF. Let u be a spectral multiplicity function on a class M of positive finite Borel measures and let $\{\mu^k\}$, $\{\underline{\mu}^k\}$ be the canonical representations of u on (H, E^t, G), $(\underline{H}, \underline{E}^t, G)$ respectively. Since (H, E^t, G) and $(\underline{H}, \underline{E}^t, G)$ are uniform resolution spaces, Theorem 1 implies that $\{\mu^k\}$ $(\{\underline{\mu}^k\})$ consists of a single Borel measure μ^k $(\underline{\mu}^k)$ such that μ^k $(\underline{\mu}^k)$ is equivalent to the Haar measure m $(\underline{m})$. Since any two Haar measures m, $\underline{m}$ on a locally compact group G are equivalent ([6 : p. 263]), μ^k and $\underline{\mu}^k$ also equivalent whence $u(\mu^k) = u(\underline{\mu}^k)$. Thus the equality $c_1 = c_2$ is an immediate consequence of the equalities $c_1 = u(\mu^k)$, $c_2 = u(\underline{\mu}^k)$. This completes the proof.

As an example on a finite dimensional space, let $H = R^3$ and let T on R^3 be given by

$$T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then the unitary extension U on R^4 of T is given by

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Since the eigenvalues of T (U) are $\lambda_1 = 0$ and $\lambda_2 = 1$ with multiplicity two ($\underline{\lambda}_1 = 1$ with multiplicity 3 and $\lambda_2 = -1$) and since spectral multiplicity function on a finite dimensional space is the minimum value of multiplicities of eigenvalues ([7: pp. 84-85]) $u(\lambda_1 = 1, \lambda_2 = 0) = 1$, $u(\underline{\lambda}_1 = 1, \underline{\lambda}_2 = -1) = 1$, and so $c_1 = c_2 = 1$ in this case. Moreover the minimal extension space $\underline{H}$ is given by

$$\underline{H} = \bigcup_{n \in Z} \{U^n h : h \in H\} = H \oplus UH = R^3 \oplus R \cong R^4.$$

Theorem 2 and the structure of minimal extension resolution space lead us to

THEOREM 3. *Let* (H, E^t, G) *and* $(\underline{H}, \underline{E}^t, G)$ *be defined as before. If* T *is a contraction on* (H, E^t, G), U *its unitary extension on* $(\underline{H}, \underline{E}^t, G)$, *then*

$$L_2(\underline{m}, \underline{H}) = \bigcup_{n\in Z}\{U^n x(t) : x(t)\in L_2(m, H),\ t\in G\}.$$

PROOF. In view of Sz.-Nagy ([10]), Sz.-Nagy and Foias ([11]) $\underline{H}$ is generated by the set $\bigcup_{n\in Z}\{U^n x(t) : x(t)\in L_2(m, H),\ t\in G\}$. Since U is unitary, $\|U^n x\|_{\underline{H}} = \|x\|_H$ for every n in Z. Thus the theorem is a consequence of the following equalities:

$$\begin{aligned} L_2(\underline{m}, \underline{H}) &= \bigcup_{n\in Z}\{U^n x(t) : x(t)\in H,\ \|U^n x\|^2_{\underline{H}} = \int_G \|U^n x(t)\|^2_{\underline{H}}\, dm(t) < \infty\} \\ &= \bigcup_{n\in Z}\{U^n x(t) : x(t)\in H,\ \|x\|^2_H = \int_G \|x(t)\|^2_H\, dm(t) < \infty\} \\ &= \bigcup_{n\in Z}\{U^n x(t) : x(t)\in L_2(m, H),\ t\in G\} \end{aligned}$$

thereby proving the theorem.

Suppose now T is in addition time-invariant (that is, it commutes with unitary operators) on a uniform resolution space (H, E^t, G) and suppose $(\underline{H}, \underline{E}^t, G)$ is any uniform resolution space. Saeks showed the following theorem ([1:pp. 329-333]):

THEOREM 4. *There is a time-invariant unitary extension* U *on* $(H\oplus\underline{H}, E^t\oplus\underline{E}^t, G)$ *of* T *on* (H, E^t, G) *such that*

$$U = \begin{pmatrix} T & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \qquad (3)$$

where $U_{12}=0$, U_{21} *and* U_{22} *are causal.*

An application of Theorem 4, Sz.-Nagy ([10: pp. 12-26]), Sz.-Nagy and Foias ([11: pp. 16-19]) allows us to extend Theorem 4 to an infinite direct sum of uniform resolution spaces, namely,

THEOREM 5. *Let* T *be a contraction on a uniform resolution space* (H, E^t, G) *and let* (H_n, E^t_n, G) *be any uniform resolution space for each* $n=0, \pm1, \pm2, \cdots$ where $H_0=H$, $E^t_0=E^t$. *Then the representation of unitary extension* U *on* $(\sum_{n\in Z}+H_n, \sum_{n\in Z}+E^t_n, G)$ *of* T *is given by*

$$U = \begin{pmatrix} \ddots &&&&&&&&& \\ & I &&&&&&&& \\ && I &&&&&&& \\ &&& I &&&&&& \\ &&&& T & U_{12} &&&& \\ &&&& U_{21} & U_{22} &&&& \\ &&&&&& I &&& \\ &&&&&&& I && \\ &&&&&&&& I & \\ &&&&&&&&& \ddots \end{pmatrix} \tag{4}$$

If in addition T is time-invariant, all the components of U except T are causal.

Note that if in particular $H_n=H$ for each $n=0, \pm1, \pm2, \cdots$, the representation (4) is reduced to Sz.-Nagy's for $n=0, 1, 2, \cdots$, ([10:pp. 12-26]), Sz.-Nagy and Foias's for $n=0, \pm1, \pm2, \cdots$, ([11: pp. 16-19]).

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