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TOPOLOGIES ON GENERALIZED SEMI-INNER PRODUCT ALGEBRAS, LATTICES, AND SPACES

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1. Introduction

In his paper [2], Lumer has introduced the concept of semi-inner product space. This concept has led Husain and Malviya [1] to introduce and study, what they call, semi-inner product algebras. In a different direction Nath [4] has used Lumer's concept to introduce, what he calls, generalized semi-inner product spaces, and has studied strong topologies on such spaces. These strong topologies have earlier been studied on generalized inner product spaces by Prugovecki [6].

In this paper we introduce, what we call, generalized semi-inner product (in short g.s.i.p.) algebras, and lattices, and show that a g.s.i.p. algebra (lattice) with strong topology is locally convex algebra (lattice). We also show that a g.s.i.p. algebra with strong topology under a restriction is locally m-convex algebra. Finally, we show that it is possible to introduce, as in [6], weak topologies in g.s.i.p. spaces, and it turns out that a g.s.i.p. space with weak topology is a Hausdorff locally convex space.

Throughout in this paper, we have used $\mathbf{N}$ to denote the set of natural numbers.

2. g.s.i.p. algebras

G. Lumer [2] calls a complex (real) vector space X a semi-inner product space (abbreviated to s.i.p. space) if to every pair of of elements $x, y \in X$ there corresponds a complex (real) number, written as $[x,y]$, with the following properties:

(i) $[x+y, z] = [x, z] + [y, z]$,

$[\lambda x, y] = \lambda [x,y], x, y, z \in X$, λ: complex (real),

(ii) $[x, x] > 0$ for $x \neq 0$, and

(iii) $|[x,y]|^2 \leq [x,x][y,y]$.

REMARK 2.1. With $\|x\| = [x,x]^{1/2}$, a s.i.p. space becomes a normed space.

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It is clear from (i) and (ii) that $[x, y]=0$ for all $y\in X$ iff $x=0$, Moreover, if either of x,y is zero, then $[x,y]=0$.

DEFINITION 2.2. A vector space A is called a *s.i.p. algebra* if (a) A is a normed algebra, and (b) A is a s.i.p. space with the same norm as that of the normed algebra A,

REMARK 2.3. Our definition of a s.i.p. algebra is different from that given in [1].

DEFINITON 2.4. A vector space A is called *a generalized semi-inner product algebra* (abbreviated to g.s.i.p. algebra) if

(i) A is an algebra,

(ii) there is a subspace M of A which is a s.i.p. algebra, and

(iii) there is a set ζ of linear multiplicative operators on A satisfying (a) $\zeta A\subset M$, i.e. each member of ζ maps A into M, and (b) $Tx=0$ for all $T\in\zeta$, implies $x=0$.

We denote such a g.s.i.p. algebra by the triple (A,ζ,M).

EXAMPLE 2.5. Let A be the space of all measurable functions on a compact topological group G with Haar measure. A is a vector space if

$$(f+g)(x)=f(x)+f(y)$$
$$\text{and } (\lambda f)(x)=\lambda f(x),\ f,\ g\in A,\ \lambda \text{ scalar.}$$

Clearly A is an algebra. Now, consider the vector subspace $M=L^p(G), 2\leq p<\infty$, of A; then M is a s.i.p. algebra if

$$(fg)(x)=\int_G f(xy^{-1})g(y)dy$$

and

$$[f,\ g]=\frac{1}{\|g\|_p^{p-2}}\int_G f(x)|g(x)|^{p-1}\ \mathrm{sgn}\ (g(x))\ dx,$$

where

$$\|g\|_p=\left(\int_G |g(x)|^p\ dx\right)^{1/p}$$

and sgn is the signum function. Let ζ be the family of operators $E^{p-1}(S)$ defined by

$$(E^{p-1}(S)(\alpha f+\beta g))(x)=\chi_S(x)(|f(x)|^{p-2}+|g(x)|^{p-2})(\alpha f(x)+\beta g(x)),$$

where χ_S is the characteristic function of S and S is a subset of A whose Haar measure is non-zero. Clearly each $E^{p-1}(S)$ is linear. It can easily be verified that

$\zeta A \subset M$ and $Tx=0$ for all $T\in\zeta$ implies $x=0$. Since

$$\chi_S=\chi_S\chi_S,$$

it follows that each $E^{p-1}(S)$ is multiplicative. Thus, (A, ζ, M) is a g.s.i.p. algebra. But it is not a s.i.p. algebra.

3. Strong topology

Let (A, ζ, M) be a g.s.i.p. algebra. We introduce "strong topology" in A as follows:

DEFINITION 3.1. For each $x\in A$, define

$$V(x;\ T_1,\ \cdots,\ T_n;\ \varepsilon)=\{y\in A;\ [T_i(y-x),\ T_i(y-x)]^{1/2}<\varepsilon,\ 1\leq i\leq n\}$$

for all $\varepsilon>0$, $T_1, \cdots, T_n\in\zeta$ and $n\in N$. The family $\{V(x;\ T_1,\ \cdots,\ T_n;\ \varepsilon)\colon T_1, \cdots, T_n\in\zeta,\ \varepsilon>0,\ n\in N\}$ forms a neighbourhood basis at x, for each $x\in A$, for a topology on A which we call "strong topology".

REMARK 3.2. It is known, from [4], that (i) each $V(0;\ T_1,\ \cdots,\ T_n;\ \varepsilon)$ is circled and convex, and that (ii) the topology on A for which the sets $V(x;\ T;\ \varepsilon)$ are neighbourhoods of x for all $\varepsilon>0$, $T\in\zeta$ is Hausdorff.

Michael [3] calls a subset of an algebra m-convex (multiplicatively convex if V is convex and idempotent (i.e. $VV\subset V$).

LEMMA 3.3. *Let (A, ζ, M) be a g.s.i.p. algebra. Then each $V(0;\ T_1,\ \cdots,\ T_n;\ \varepsilon)$, $0<\varepsilon\leq 1$, is m-convex.*

PROOF. Clearly $V(0;\ T_1,\ \cdots,\ T_n;\ \varepsilon)$ is convex by 3.2. We show that it is idempotent i.e. $V(0;\ T_1,\ \cdots,\ T_n;\ \varepsilon)V(0;\ T_1,\ \cdots,\ T_n;\ \varepsilon)\subset V(0;\ T_1,\cdots,T_n;\ \varepsilon)$ $(0<\varepsilon\leq 1)$. Let $x,\ y\in V(0;\ T_1,\cdots,\ T_n;\ \varepsilon)$. Then,

$$\begin{aligned}[T_k(xy),\ T_k(xy)]^{1/2}&=[T_k(x)T_k(y),\ T_k(x)T_k(y)]^{1/2},\quad 1\leq k<n\\ &\leq[T_k(x),\ T_k(x)]^{1/2}[T_k(y),\ \ T_k(y)]^{1/2}<\varepsilon^2\leq\varepsilon.\end{aligned}$$

This implies that $xy\in V(0;\ T_1,\ \cdots,\ T_n;\ \varepsilon)$ for all $x,\ y\in V(0;\ T_1,\ \cdots,\ T_n;\ \varepsilon)$, $0<\varepsilon\leq 1$, and this establishes the idempotentness of $V(0;\ T_1,\ \cdots,\ T_n;\ \varepsilon)$, $0<\varepsilon\leq 1$. Hence it is m-convex.

A locally convex algebra is an algebra and a Hausdorff locally convex space such that the multiplication is continuous in each variable separately ([3], page 3). A locally convex algebra is called locally m-convex algebra if there exists a neighbourhood basis of 0 consisting of m-convex sets ([3], page 6).

THEOREM 3.4. *A g.s.i.p. algebra (A, ζ, M) equipped with the strong topology, as defined in* 3.1, *is a locally convex algebra.*

PROOF. In view of ([4], 3.3), (A, ζ, M) is a Hausdorff locally convex space. To complete the proof we show that for any $V(x_0x;\ T_1, \cdots, T_n;\ \varepsilon)$, there exists

$$V\left(x;\ T_1, \cdots, T_n;\ \frac{\varepsilon}{\lambda}\right),\ \lambda=\max_{1\leq k\leq n}(\lambda_k),$$

$$\lambda_k=[T_k(x_0),\ T_k(x_0)]^{1/2},\ k=1, \cdots, n, \text{ such that}$$

$$x_0V\left(x;\ T_1, \cdots, T_n;\ \frac{\varepsilon}{\lambda}\right)\subset V(x_0x;\ T_1, \cdots, T_n;\ \varepsilon).$$

Let $y\in V\left(x;\ T_1, \cdots, T_n;\ \frac{\varepsilon}{\lambda}\right)$; then

$$[T_k(y-x),\ T_k(y-x)]^{1/2}<\frac{\varepsilon}{\lambda},\ k=1, 2, \cdots, n.$$

Now,

$$\begin{aligned}[T_k(x_0y-x_0x),\ T_k(x_0y-x_0x)]^{1/2}&=[T_kx_0(y-x),\ T_kx_0(y-x)]^{1/2}\\ &=[T_k(x_0)T_k(y-x),\ T_k(x_0)T_k(y-x)]^{1/2}\\ &\leq[T_k(x_0), T_k(x_0)]^{1/2}[T_k(y-x),\ T_k(y-x)]^{1/2}\\ &<\lambda\frac{\varepsilon}{\lambda}=\varepsilon.\end{aligned}$$

This implies that $x_0y\in V(x_0x;\ T_1, \cdots, T_n;\ \varepsilon)$, for all $y\in V(x;\ T_1, \cdots, T_n;\ \varepsilon/\lambda)$, and this proves that

$$x_0V(x;\ T_1, \cdots, T_n;\ \varepsilon/\lambda)\subset V(x_0x;\ T_1, \cdots, T_n;\ \varepsilon).$$

Similarly, we can show that

$$V(x;\ T_1, \cdots, T_n;\ \varepsilon/\lambda)x_0\subset V(xx_0;\ T_1, \cdots, T_n;\ \varepsilon).$$

This shows that (A, ζ, M) is a locally convex algebra under the strong topology.

COROLLARY 3.5. *A g.s.i.p. algebra (A, ζ, M) equipped with the strong topology for which the family $\{V(x:\ T_1, \cdots, T_n;\ \varepsilon);\ T_1, \cdots, T_n\in\zeta,\ 0<\varepsilon\leq 1,\ n\in N\}$ forms a neighbourhood basis of x, for each $x\in A$, is a locally m-convex algebra.*

PROOF. The result follows from 3.3 and 3.4 if λ, as defined in 3.4, is greater than or equal to 1. If $\lambda<1$ then the result follows from 3.3, because we can show that for any $V(x_0x;\ T_1, \cdots, T_n;\ \varepsilon)$, there exists $V(x;\ T_1, \cdots, T_n;\ \varepsilon)$ such that

$$x_0V(x;\ T_1, \cdots, T_n;\ \varepsilon)\subset V(x_0x;\ T_1, \cdots, T_n;\ \varepsilon).$$

THEOREM 3.6. *A g.s.i.p. algebra (A, ζ, M) with strong topology is metrizable* *if there is a countable subset C of ζ which has the proberty that for any $T\in\zeta$ there is a $P\in N$, where N is the linear manifold generated by C, such that $[Tx, Tx]^{1/2} \leq [Px, Px]^{1/2}$ for all $x\in A$.*

PROOF. Same as that of ([4], 3.4).

4. g.s.i.p. lattices

In this section, we consider only the *real* vector spaces.

DEFINITION 4.1. An ordered vector space X is called *a semi-inner product lattice* (in short s.i.p. lattice) if (a) X is a normed lattice, and (b) X is a s.i.p. space with the same norm as that of the normed lattice X.

A subset B of a vector lattice X is solid if $|x|\leq|y|$, $y\in B$ implies $x\in B$, where $|x|=\sup\{-x, x\}$. A vector subspace M of X is a lattice ideal if M is a solid subset of X. Every lattice ideal M in a vector lattice X is a sublattice of X ([5], page 35). Every solid set is circled.

A linear map from a vector lattice into a vector lattice is called a lattice homomorphism if it preserves lattice operations. (For more details, see [5]).

DEFINITION 4.2. An ordered vector space X is called *a generalized semi-inner product* (in short g.s.i.p.) lattice if

(i) X is a vector lattice,

(ii) there is a lattice ideal M of X which is a s.i.p. lattice, and

(iii) there is a set τ of lattice homomorphisms on X such that (a) $\tau X\subset M$, i.e. each member of τ maps X into M, and (b) $hx=0$ for all $h\in\tau$ implies $x=0$.

We denote such a g.s.i.p. lattice by the triple (X, τ, M). The example given in [4] is in fact a g.s.i.p. lattice.

As in section 2, we define

$V(x;\ h_1, \cdots, h_n;\ \varepsilon)=\{y\in X;\ [h_i(y-x), h_i(y-x)]^{1/2}<\varepsilon,\ 1\leq i\leq n\}$, for all $\varepsilon>0$, $h_1, \cdots, h_n\in\tau$ and $n\in N$.

LEMMA 4.3. *Each $V(0;\ h_1, \cdots, h_n;\ \varepsilon)$ is solid.*

PROOF. Let $|x|\leq|y|$ and $y\in V(0;\ h_1, \cdots, h_n;\ \varepsilon)$. Then (*) $[h_i(y), h_i(y)]^{1/2}<\varepsilon$, $1\leq i\leq n$. Since each h_i is a lattice homomorphism, $|x|\leq|y|$ implies that $|h_i(x)|\leq |h_i(y)|$. But then, since M is a normed lattice, we have

$$\|h_i(x)\| \leq \|h_i(y)\|$$

i.e. $$[h_i(x),\ h_i(x)]^{1/2} \leq [h_i(y), h_i(y)]^{1/2} < \varepsilon, \text{ by } (*).$$

This implies that $x \in V(0;\ h_1,\ \cdots,\ h_n;\ \varepsilon)$, and hence each $V(0;\ h_1,\ \cdots,\ h_n;\ \varepsilon)$ is solid.

An ordered locally convex space which is a vector lattice is called a locally convex lattice if there is a neighbourhood basis of 0 consisting of solid sets ([5], page 103).

THEOREM 4.4. *A g.s.i.p. lattice (X, τ, M) equipped wih the strong topology for which the family $\{V(x;\ h_1,\ \cdots,\ h_n;\ \varepsilon):\ \varepsilon>0,\ h_1,\ \cdots,\ h_n \in \tau,\ n \in N\}$ forms a neighbourhood basis at x, for each $x \in X$, is a locally convex lattice.*

PROOF. In view of ([4], 3.3), (X, τ, M) is a Hausdorff locally convex space; hence it is an ordered locally convex space, because X is an ordered vector space ([5], page 63). The result now follows from 4.3.

The sets of the form

$$V(x;\ h_1,\ h_2,\ \cdots;\ \varepsilon) = \bigcap_{k=1}^{\infty} V(x;\ h_k;\ \varepsilon),\ h_k \in \tau,\ \varepsilon>0,$$

constitute a neighbourhood basis at x, for each $x \in X$, for a topology on X called ultra-strong topology.

Clearly ultra-strong topology is finer than the strong topology, and hence Hausdorff.

REMARK 4.5. Since the intersection of solid sets is solid, it follows that each $V(0;\ h_1,\ h_2,\ \cdots;\ \varepsilon)$ is solid. Also it is convex.

THEOREM 4.6. *A g.s.i.p. lattice (X, τ, M) equipped with ultra-strong topology is a locally convex lattice.*

PROOF. (x, τ, M) is a Hausdorff locally convex space [4], and hence ordered locally convex space. The result now follows from 4.5.

5. Weak topology

A vector space X is a g.s.i.p. space if

(i) there is a subspace M of X which is a s.i.p. space,

(ii) there is a (non-empty) set ζ of linear operations on X satisfying (a) $\zeta X \subset M$, i.e. each element of ζ maps X into M, and (b) $Tx=0$ for all $T \in \zeta$ implies $x=0$ [4].

LEMMA 5.1. *Let (X, ζ, M) be a g.s.i.p. space and $x \in X$. If $[Tx, x]=0$ for* *all $y \in M, T \in \zeta$, then $x=0$.*

PROOF. In view of (ii) (b), we have $Tx=0$ for all $T \in \zeta$. But then, by 2.4, it follows that $x=0$.

LEMMA 5.2. *Let (X, ζ, M) be a g.s.i.p. space and $x \in X$. If $[Tx, Tx]=0$* for all $T \in \zeta$, then $x=0$.

PROOF. First we observe that $Tx=0$; because if $Tx \neq 0$, then by the definition of semi-inner product Space, it follows that $[Tx, Tx]>0$ which contradicts the hypothesis. Since $Tx=0$ for all $T \in \zeta$, it folows from (ii)(b) that $x=0$.

DUAL SPACE 5.3. Let (X, ζ, M) be a g.s.i.p. space. For each $T \in \zeta$ and each $y \in M$, we define.

$$F(x;\, T,\, y)=[Tx,\, y] \text{ on } X.$$

Clearly F is linear functional on X, Let L_0 be the family of all such functionals; it is not a vector space, in general. Denote by L the vector space (over the same field as that of X) spanned by L_0.

PROPOSITION 5.4. *L and X constitute a dual pair.*

PROOF. If $F(x)=0$ for all $F \in L$, then $[Tx, y]=0$ for all $y \in M$ and all $T \in \zeta$. But then, in view of 5.1, $x=0$.

Conversely if for a given $F_0 \in L$ we have that $F_0(x)=0$ for all $x \in X$; then, by definition, F_0 is the zero element of L.

NOTATION 5.5. $<x, F>=F(x)$, $x \in X$, $F \in L$. Clearly $<x, F>$ is a bilinear function on X and L.

PROPOSITION 5.6. *Each $F \in L$ is continuous on X in the strong topology.*

PROOF. Let $\varepsilon>0$.

$$\begin{aligned}|F(x;\, T,\, y)-F(x_0;\, T,\, y)| &= |[Tx,\, y]-[Tx_0,\, y]| \\ &= |[T(x-x_0),\, y]| \\ &\leq [T(x-x_0),\, T(x-x_0)]^{1/2}[y,\, y]^{1/2}<\varepsilon\end{aligned}$$

whenever

$$[T(x-x_0),\, T(x-x_0)]^{1/2}<\frac{\varepsilon}{[y,y]^{1/2}},$$

because $[y, y]>0$ for $y \neq 0$, i.e. for all $x \in V\left(x_0;\, T, \frac{\varepsilon}{[y,\, y]^{1/2}}\right)$. Thus each element of L is a continuous linear functional on E equipped with strong topology.

Hence the continuity of an arbitrary element of L follows.

REMARK 5.7. The above proposition says that L is contained in the vector space conjugate to X with strong topology.

DEFINITION 5.8. The coarsest topology on X for which all the linear functionals from L are continuous is called *the weak topology.*

The family of all subsets of X of the form,

$$U(x\ ;F_1, \cdots, F_n)=\{y\in X: |F_i(y-x)|<1,\ 1\leq i\leq n\}$$

for all $F_1, \cdots, F_n\in L$, $n\in N$, is a neighbourhood basis at x.

Since L_0 generates L, the family of all neighbourhoods,

$$U(0;\ y_1,\ T_1, \cdots, y_n,\ T_n)=\{x\epsilon X: |[T_ix,\ y_i]|<1,\ 1\leq i\leq n\}$$

corresponding to all $y_1, \cdots, y_n\in M$, $T_1, \cdots, T_n\in\zeta$, $n=1.2\cdots$, is also a neighbourhood basis of 0.

As X and L are dual pairs, X is a Hausdorff topological space in the weak topology.

From the general properties of weak topologies we have the following result:

PROPOSITION 5.9. *A g.s.i.p. space (X, ζ, M) is a Hausdorff locally convex space in the weak topology.*

DEFINITION 5.10. The sets of the form

$$U(x;\ F_1, \cdots, F_n, \cdots)=\{y\in X: |F_i(y-x)|<1,\ i=1,\ 2, \cdots, n, \cdots\}$$

for all sequences $F_1, F_2, \cdots, F_n, \cdots\in\zeta$, contsitute a neighbourhood basis at x, for each $x\in X$, for a topology on X which we call infra-weak topology.

Cleary infra-weak topology is finer than the weak topology and hence Hausdorff. Also each $U(x;\ F_1, \cdots, F_n, \cdots)$ is convex. It is a routine matter to establish the truth of the following result.

PROPOSITION 5.11. *A g.s.i.p. space (X, ζ, M) with the infra-weak topology is a Hausdorff locally convex space.*

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REFERENCES

[1] T. Husain and B.D. Malviya, *On semi-inner product spaces II*, Colloquium Mathematics, vol. XXVII, Fasc. 1 (1973), 95-105.

[2] G. Lumer, *Semi-inner product spaces*, Transactions of the A.M.S. 100 (1961), 29-43.

[3] E.A. Michael, *Locally multiplicatively-convex topological algebras*, Memoirs of the A.M.S., no. 11 (1952).

[4] B. Nath, *Topologies on generalized semi-inner product spaces*, Compositio Mathematica, vol. 23, Fasc. 3, (1971), 309-316.

[5] A.L. Peressini, *Ordered toporogical vector spaces*, Harper & Row, New York (1967).

[6] E. Prugovecki, *Topologies on generalized inner product spaces*, Canadian J. Math. 21 (1969), 158-169.