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A STUDY ON TOPOLOGICAL MULTI-SEMIGROUPS

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The Study of discrete set-valued multiplications on a set is originated and developed by O. Ore. On the other hand, the topological observations of set-valued functions have been investigated extensively over the past forty years. The author developed a basic theory of set-valued topological algebra [1] combining the above two algebraic and topological concepts together.

This paper is devoted to the investigation of multi-semigroup multiplications on an interval. It is shown that a multi-semilattice on an interval in which an end point is a zero has the exact structure of a topological semilattice. Also some other properties of multi-semigroups are studied in terms of usual standard threads.

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1. Introduction

Let X be a space and let 2^X be the set of all non-void closed subsets of X. With each subset A of X we associate the following subsets of 2^X:

$$L(A)=\{B\in 2^X \mid B\subset A\},\ \ M(A)=\{B\in 2^X \mid B\cap A\neq\square\}.$$

Throughout this paper, all spaces in consideration are assumed to be Hausdorff and 2^X is assumed to have the Vietoris topology having the family

$$\{L(U)\mid U=U^0\subset X\}\cup\{M(V)\mid V=V^0\subset X\}$$

as a subbase of it [2].

DEFINITION 1.1. A ***multi-semigroup*** is a nonvoid Hausdorff space S together with a continuous function

$$S\times S\longrightarrow 2^S$$

(whose value at (x, y) will be denoted by xy) satisfying

$$(xy)z=x(yz)$$

for all x, y, z in S. Here, AB is defined to denote the union $\bigcup\{ab\mid a\in A,\ b\in B\}$ *for* $A, B\subset S$.

The proof of the following lemma may be found in [1].

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LEMMA 1.2. *Let A and B be compact subsets of the multi-semigroup S. If AB is contained in an open subset W of S, then there exist open subsets U and V of S such that*

$$A\subset U,\ B\subset V,\ \text{and } UV\subset W.$$

By using the above lemma, one may obtain

LEMMA 1.3. *Let S be a multi-semigroup. If A is compact and if B is open in S, then the set*

$$\{x\in S\mid Ax\subset B\}$$

is open in S.

The following theorem, for a semigroup, is due to A. D. Wallace and J. M. Day and appears in [7]. By the aid of lemma 1.3, the same theorem holds for a multi-semigroup.

THEOREM 1.4. *Suppose S is a continuum multi-semigroup. If H is a subset of S with nonempty boundary $F(H)$ and if H^* contains a point t such that $St\subset H^*$, then $Sb\subset H^*$ for some b in $F(H)$.*

DEFINITION 1.5. Let S be a multi-semigroup.

(1) An element s of S is called a *left scalar* if and only if sx is a singleton for each x in S.

(2) An element u of S is called a *left unit(left scalar unit)* if and only $x\in ux$ $(x=ux)$ for each x in S.

(3) An element e of S is called an *idempotent(multi-idempotent)* if and only if $e^2=e$ $(e\in e^2)$.

(4) An element f of S is called a *left scalar idempotent* if and only if f is a left scalar and an idempotent.

In ench definition above, right and two-sided elements are defined analogously.

CONVENTIONS Throughout, $I=[a,\ b]$ will denote the real closed interval from a to b and a semigroup will always mean a topological semigroup([6], [7]).

DEFINTION 1.6. A subset A of a multi-semigroup S is called a *left(right, two-sided) ideal* of S if and only if

$$SA\subset A\ (AS\subset A,\ SA\cup AS\subset A).$$

As an immediate application to theorem 1.4, we have

COROLLARY 1.7. *Suppose I is a multi-semigroup in which a is a zero. Then*

(1) *$[a,\ x]$ is an ideal of I for each x in I.*

(2) *If I has a unit, then $Ix=[a, x]=xI$ for each x in I.*

(3) *If e is a multi-idempotent of I, then*

$$Ie=[a, e]=eI.$$

LEMMA 1.8. *If $f, g: 2^I \longrightarrow I$ are functions defined by*

$$f(A)=\inf A, \qquad g(A)=\sup A$$

then f and g are continuous.

PROOF. Let $A \in 2^I$ and let $U=(c, d)$, the open interval from c to d such that $f(A)=x \in U$. Let $V=(c, b]$. Then $A \in L(V) \cap M(U)$. If $B \in L(V) \cap M(U)$, then $B \subset V$ and $B \cap U \neq \square$. Therefore $c < f(B) < d$, i.e., $f(B) \in U$. Hence f is continuous. Similarly, g is continuous.

2. Standard multi-semigroups

DEFINITION 2.1. A multi-semigroup on I will be called a *standard multi-semigroup* if and only if a is a zero and b is a scalar unit. For the definition of a standard thread in the theory of semigroups, see [4] and [5].

LEMMA 2.2. *Suppose I is a multi-semigroup in which a is a zero. If e is an idempotent of I, then $[a, e]$ is a standard multi-semigroup. In particular, I is a standard multi-semigroup if b is an idempotent.*

PROOF For each x in $[a, e]$, define $x'=\inf (ex)$. In view of (1) in Corollary 1.7, $(x')' \leq x' \leq x$ for each x in $[a, e]$. Note that $x' \in ex$ for each x in $[a, e]$ since ex is closed. Now since $ex=e(ex)=\bigcup \{ey \mid y \in ex\}$, $ey \subset ex$ for each $y \in ex$. It follows that $ex' \subset ex$, and

$$(x')'=\inf (ex') \geq \inf (ex)=x' \geq (x')',$$

i.e., $(x')'=x'$. Define a function $f: [a, e] \longrightarrow [a, e]$, via $f(x)=x'$. Then f is continuous by Lemma 1.8. Morover,

$$f^2(x)=f(f(x))=f(x')=(x')'=x'=f(x),$$

i.e., $f^2=f$ and f is a retraction. Since $f(a)=a'=a$ and $f(e)=e'=e$, f is a surjection. Hence $f(x)=x$, i.e., $ex=x$ for each x in $[a, e]$. Similarly, $xe=x$ for each x in $[a, e]$ so that e is a scalar unit for $[a, e]$. By (1) in Corollary 1.7, $[a, e]$ is a standard multi-semigroup.

CONVENTION For a multi-semigroup on I, the following notation will be adopted throughout the remainder of this paper. For each x and each y in I, denote

$$x \wedge y=\inf (xy), \quad x \vee y=\sup (xy).$$

LEMMA 2.3. *Let I be a standard multi-semigroup and let $x, y, u, v \in I$ with $x \leq y$*

and $u \leq v$. *Then*

$$x \vee u \leq y \vee v.$$

PROOF. Since $x \leq y$, $x \in [a, y] = Iy$. Then

$$xu \subset (Iy)u = I(yu) = \bigcup \{It \mid t \in yu\} = \bigcup [a, t] \mid t \in yu\} = [a, y \vee u].$$

It follows that $x \vee u \leq y \vee u$. In a similar way, $y \vee u \leq y \vee v$ may be established. Therefore $x \vee u \leq y \vee u \leq y \vee v$.

THEOREM. 2.4. *If I is a standard multi-semigroup, then $(I, \vee)$ is a standard thread.*

PROOF. Let $x, y, z \in I$. Since $x \vee y \in xy$, $(x \vee y)z \subset (xy)z = xyz$, i.e.,

$$(x \vee y) \vee z \leq \sup (xyz) = \sup (\bigcup \{tz \mid t \in xy\}) = \sup \{t \vee z \mid t \in xy\}.$$

Since $t \leq x \vee y$ for every $t \in xy$, by Lemma 2.3, $t \vee z \leq (x \vee y) \vee z$ for all t in xy. It follows that

$$(x \vee y) \vee z \leq \sup (xyz) = \sup \{t \vee z \mid t \in xy\} \leq (x \vee y) \vee z,$$

and $(x \vee y) \vee z = \sup (xyz)$. Similarly, $x \vee (y \vee z) = \sup (xyz)$, i.e.,

$$(x \vee y) \vee z = \sup (xyz) = x \vee (y \vee z).$$

LEMMA 2.5. *Let I be a standard multi-semigroup such that $x \wedge z \neq y \wedge z$ for all $x, y, z \in I$ with $x < y$ and $z \neq a$. Then $x \leq y$ implies $x \wedge z \leq y \wedge z$ for all $z \in I$.*

PROOF. Let $u < v$ in I and let

$$A = \{z \in (a, b] \mid u \wedge z < v \wedge z\}.$$

Then $A \neq \square$ since $b \in A$. If $z_0 \in A$, then $u \wedge z_0 < v \wedge z_0$. Pick a point t so that $u \wedge z_0 < t < v \wedge z_0$. By the continuity of the operation $\wedge$, there is an open set W about z_0 such that $\{u \wedge w \mid w \in W\} \subset [a, t)$ and $\{v \wedge w \mid w \in W\} \subset (t, b]$, i.e., $W \subset A$. Therefore A is an open subset of $(a, b]$. By hypothesis, $(a, b] - A = \{z \in (a, b] \mid u \wedge z > v \wedge z\}$. In a similar way, it can be also shown that $(a, b] - A$ is open. Then A is a proper clopen subset of $(a, b]$ if $(a, b] - A$ is nonvoid. Therefore $A = (a, b]$.

THEOREM 2.6. *Suppose I is a standard multi-semigroup such that $x \wedge z \neq y \wedge z$ for all x, y $z \in I$ with $x < y$ and $z \neq a$. Then $(I, \wedge)$ is a standard thread.*

PROOF. Let $x, y, z \in I$. Since $x \wedge y \in xy$, $(x \wedge y)z \subset (xy)z = xyz$. Hence

$$(x \wedge y) \wedge z \geq \inf (xyz) = \inf (\bigcup \{tz \mid t \in xy\}) = \inf \{t \wedge z \mid t \in xy\}.$$

Since $t \geq x \wedge y$ for every $t \in xy$, by Lemma 2.5, $t \wedge z \geq (x \wedge y) \wedge z$ for all t in xy. It follows that

$$(x \wedge y) \wedge z \geq \inf (xyz) = \inf \{t \wedge z \mid t \in xy\} \geq (x \wedge y) \wedge z,$$

i.e., $(x \wedge y) \wedge z = \inf (xyz)$. Similarly, $x \wedge (y \wedge z) = \inf (xyz)$.

THEOREM 2.7. *Suppose $(I, *)$ and $(I, *')$ are standard threads such that $x*y \leq x*'y$ for each $x, y \in I$. Then I is a standard multi-semigroup under the multipltiplication(denoted by juxtaposition)*

$$xy = [x*y,\ x*'y].$$

PROOF Clearly the multiplication is continuous. To show the associative law, let x, y, z be in I. Then $t_1*z \leq t_2*z$ and $t_1*'z \leq t_2*'z$ whenever $t_1 \leq t_2$. If $t \in xy$ then $x*y \leq t \leq x*'y$ so that $(x*y)*z \leq t*z$ and $t*'z \leq (x*'y)*'z$. Since tz is connected for all t in xy, $(xy)z$ is connected [2]. It follows that

$$(xy)z = [(x*y)*z,\ (x*'y)*'z] = [x*(y*z),\ x*'(y*'z)] = x(yz),$$

i.e., $(xy)z = x(yz)$. Clearly, $ax = a = xa$ and $bx = x = xb$.

3. Multi-semilattices

DEFINITION 3.1. A multi-semigroup S is said to be a *multi-band* if and only if every element is an idempotent.

A *multi-semilattice* is a commutative multi-band.

THEOREM 3.2. *If I is a multi-band in which a is a zero, then $xy = min\{x, y\}$, i.e., each such multi-band is a topological semilattice.*

PROOF. Since every element is an idempotent, by Lemma 2.2, it is readily shown that $xy = x = yx$ whenever $x \leq y$, i.e., $xy = \min\{x, y\}$.

LEMMA 3.3. *Suppose I is a multi-band. If $v \in uv$ $(u \in uv)$ for all u and v in I with $u < v$, then*

$$uv \cap (v,\ b] = \square \quad (uv \cap [a,\ u) = \square).$$

PROOF. Let u and v be in I with $u < v$. For each $x \in [u, b]$, let $u \vee x = x'$. By hypothesis, $x \leq x'$. Since $x' \in [u, b]$, $x' \leq (x')'$. Since ux is closed for each x, $ux' \subset ux$. Then $(x')' \leq x'$ so that $(x')' = x'$. Let $A = \{ux \mid x \in [u, b]\}$. Dfine the functions

$$f: [u,\ b] \longrightarrow A, \quad g: A \longrightarrow [u,\ b]$$

via $f(x) = ux$ and $g(ux) = x'$. Then $h = gf$ is continuous and $h^2 = h$. Since $h(u) = u$ and $h(b) = b$, h is a surjection. It follows that $h(x) = x$ for all $x \in [u, b]$, and hence $u \vee v = v$, *i.e.*, $uv \cap (v, b] = \square$.

As an immediate consequence to the above lemma, one may obtain the following:

THEOREM 3.4. *If I is a multi-band such that $x, y \in xy$ and $xy \cap (x, y) = \square$ for each $x, y \in I$ with $x < y$, then I is a multi-semilattice and*

$$xy = [x,\ y].$$

THEOREM 3.5. *If I is a multi-band such that $xy \cap (x, y) \neq \square$ for each $x, y \in I$ with $x < y$, then I is a multi-semilattice and*

$$xy = [x, y].$$

PROOF. Let $x, y \in I$ with $x < y$. Suppose $[x, y] - xy \neq \square$ and let $z \in [x, y] - xy$. Since z is in the open set $I - xy$, let (c, d) be the component containing z in $I - xy$. Since xy is closed, $c, d \in xy$. Then $cd \subset xy$, and $cd \cap (c, d) = \square$. This is a contdradiction. Therefore $[x, y] \subset xy$. Since $x, y \in xy$ for each $x, y \in I$, by using Theorem 3.4, $xy = [x, y]$.

In the following, some multi-semilattice operations on I, other than those that have been given, may be found. Let $a < c < b$.

$$(1)\quad xy = yx = \begin{cases} [x, y] & (x, y \in [a, c],\ x \leq y) \\ \{x, y\} & (x, y \in [c, b]) \\ [x, c] \cup \{y\} & (x \in [a, c],\ y \in [c, b]) \end{cases}$$

$$(2)\quad xy = yx = \begin{cases} [x, y] & (x, y \in [a, c],\ x \leq y) \\ \min\{x, y\} & (x, y \in [c, b]) \\ [x, c] & (x \in [a, c],\ y \in [c, b]) \end{cases}$$

$$(3)\quad xy = yx = \begin{cases} \{x, y\} & (x, y \in [a, c]) \\ \min\{x, y\} & (x, y \in [c, b]) \\ \{x, c\} & (x \in [a, c],\ y \in [c, b]) \end{cases}$$

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