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DENSE SUBSPACES OF LOCALLY CONVEX SPACES

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I. Introduction

It is well known that:

(1) if a locally convex space E contains a dense, Mackey subspace, then E is a Mackey space.

(2) If a locally convex space E contains a dense, barrelled subspace, then E is a barrelled space.

(3) If a locally convex space E contains a dense, Baire-like subspace, then E is a Baire-like space.

(4) If a locally convex space E contains a dense, unordered Baire-like subspace, then E is a unordered Baire-like space.

In this paper we prove the following properties:

(a) If a locally convex space E contains a dense, countably barrelled subspace, then E is a countably barrelled space.

(b) If a locally convex space E contains a dense, Baire subspace, then E is a Baire space.

The notation will be essentially that used by J. Horváth. If (E, F) is a dual pairing (E and F not necessarily separating points), then $\sigma(E, F)$ will denotes the topology on E of pointwise convergence on F. The polar A° of a subset A of E is the set

$$\{f \in F : \langle a, f \rangle \leq 1 \text{ for all } a \in A\}$$

The vector space of continuous linear functionals on a locally convex space E will be designated by E′.

A locally convex space is said to be barrelled if every closed, balanced, convex, absorbing subset of E is a neighborhood of 0, equivalently, if every $\sigma(E', E)$-bounded subset of E′ is equicontinuous. A locally convex space E is said to be countably barrelled if each $\sigma(E', E)$-bounded subset of E′ which is the countable union of equicontinuous subsets of E′ is itself equicontinuous.

A locally convex space E is said to be a Mackey space if every balanced, convex, $\sigma(E', E)$-Compact subset of E′ is equicontinuous.

itself.

A locally convex space E is said to be Baire space if E is the second category in

II. Main Theorems

THEOREM 1. If a locally convex space E contains a dense, counatbly barrelled subspace M, then E is countably barrelled.

Proof. Since M is a dense subspace, M′ can be canonically identified with E′, Let $B=\bigcup_{n=1}^{\infty} U_n$ any $\sigma(E', E)$-bounded subset of E′ which is the countable union of equicontinuous subsets of E′, then B is $\sigma(M', M)$-bounded subset of M′ which is the countable union of equicontinuous subsets of M′. Since the polar of B in M is $M\cap B^0$, where B^0, is the polar of B in E′ and M is countable barrelled, $M\cap B^0$ is a neighborhood of 0 in M. But then the closure of $M\cap B^0$ is a neighborhood of 0 in E because M is dense in E. Since B^0 is a convex, $\sigma(E', E)$-closed set, B^0 is closed set in original topology, hence B^0 contains the closure of $M\cap B^0$. Consequently B^0 is a neighborhood of 0 in E.

COROLLARY: The completion of a Hausdorff countable barrelled space is a countable barrelled space.

THEOREM 2. If a locally convex space E contains a dense, Baire subspace M, then E is a Baire space.

Proof: A locally convex space E is a Baire space if and only if every absorbing, balanced, closed set has an interior point.

Let A be an absorbing, balanced, closed set in E, than $A\cap M$ is an absorbing, balanced, closed set in M. By assumption, $A\cap M$ has an interior point x in M. But then the closure of $A\cap M$ is a neighborhood of x in E, because M is a dense in E. Since A contains this closure, A is itself a neighborhood of x in E. Consequently, E is a Baire.

COROLLARY: The completion of a Hausdorff Baire space is a Baire space.

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