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A Construction of Submartingales From Supermartingales

By Kang Sup Lee

Cheong Ju University, Cheong Ju, Korea

1. Introduction

Let $\{X_1, X_2, \cdots\}$ be a sequence of integrable random variables on probability space $(\Omega, \mathfrak{F}, P)$, and $\mathfrak{F}_1 \subset \mathfrak{F}_2 \subset \cdots$ an increasing sequence of sub σ-fields of $\mathfrak{F}$; X_n is assumed $\mathfrak{F}_n$-measurable.

Ash [1] showed that how to construct submartingales from other submartingales or martingales. Similarly, we have the method of submartingales from supermartingales.

2. Preliminary

Definition. $\{X_n, \mathfrak{F}_n\}$ is a *martingale* iff for all $n=1, 2, \cdots$ $E(X_{n+1}|\mathfrak{F}_n)=X_n$ a. e.
$\{X_n, \mathfrak{F}_n\}$ is a *submartingale* iff for for $n=1, 2,$ $E(X_{n+1}|\mathfrak{F}_n) \geqq X_n$ a. e.
$\{X_n, \mathfrak{F}_n\}$ is a *supermartingale* iff for all $n=1, 2,$ $E(X_{n+1}|\mathfrak{F}_n) \leqq X_n$ a. e.

The following lemma is due to Ash [1], so we omit the proof.

Lemma. (Jensen's Inequality) *Let g be a convex function from I to R, where I is open interval of reals. Let X be a random variable on $(\Omega, \mathfrak{F}, P)$, with $X(\omega) \in I$ for all ω. Assume $E(X)$ to be finite. If $\mathfrak{H}$ is a sub σ-field of $\mathfrak{F}$, then $E[g(X)|\mathfrak{H}] \geq g[E(X|\mathfrak{H})]$ a. e.*

3. Result

Theorem. *Let $\{X_n, \mathfrak{F}_n\}$ be a supermartingale, g a convex, decreasing function from R to R. If $g(X_n)$ is integrable for all n, then $\{g(X_n), \mathfrak{F}_n\}$ is a submartingale.*

Proof. By lemma, $E[g(X_{n+1})|\mathfrak{F}_n] \geqq g[E(X_{n+1}|\mathfrak{F}_n)]$ ....(1)
And by the definition of supermartingale, $E(X_{n+1}|\mathfrak{F}_n) \leqq X_n$ ....(2)
Since g is decreasing function, $g[E(X_{n+1}|\mathfrak{F}_n)] \geqq g(X_n)$. ....(3)
By (1) and (3), $E[g(X_{n+1})|\mathfrak{F}_n] \geqq g(X_n)$. This completes proof.

REFERENCE

[1] Ash, R. B. (1972) *Real Analysis and Probability* Academic Press, New York.