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Note On Infinitesimal Minimal Variation Of Anti-Invariant Compact Submanifolds Of A Kaehlerian Manifold With Vanishing Bochner Curvature Tensor

By Bong Koo Kang

Sang-Joo Agri. & Seri. Junior College, Sang-Joo, Korea

0. Introduction

Yano and Kon [3] have recently studied infinitesimal anti-invariant normal variations of submanifolds M^m of a Kaehlerian manifold M^{2m}.

On the other hand Pak and Ki [5] have investigated to infinitesimal anti-invariant minimal variations of submanifolds of a Kaehlerian manifold of constant holomorphic sectional curvature.

In the present note, we study an infinitesimal anti-invariant minimal variation of submanifold M^m of a Kaehlerian manifold with vanishing Bochner curvature tensor. (See Thm. 3.1.)

1. Preliminaries [3]

Let M^{2m} be a real 2m-dimensional Kaehlerian manifold covered by a system of coordinate neighborhoods $\{U; x^h\}$ and almost complex structure tensor F_i^h and Hermitian metric tensor g_{ji}, where here and in the sequel, the indices h, i, j, k, ⋯ run over the range $\{\bar{1}, \bar{2}, \cdots \overline{2m}\}$. Then we have

(1.1) $$F_j^i F_i^h = -\delta_j^h, \quad F_j^t F_i^s g_{ts} = g_{ji}, \quad \nabla_j F_i^h = 0,$$

where ∇_j denotes the operator of covariant differentiation with respect to the Christoffel symbolds Γ_{ji}^h formed with g_{ji}.

Let M^n be an n-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; y^a\}$ and with metric tensor g_{cb}, where here and in the sequel, the indices a, b, c, ⋯ run over the range $\{1, 2, \cdots, n\}$.

We assume that M^n is isometrically immersed in M^{2m} by the immersion i: $M^n \longrightarrow M^{2m}$ and identify $i(M^n)$ with M^n. We represent the immersion i: $M^n \longrightarrow M^{2m}$ locally by

(1.2) $$x^h = x^h(y^a)$$

and put

(1.3) $$B_b^h = \partial_b x^h, \quad (\partial_b = \partial/\partial y^b),$$

which are n linearly independent vectors of M^{2m} tangent to M^n.

Since the immersion i is isometric, we have

(1.4) $$g_{ji}B_c{}^jB_b{}^i=g_{cb}.$$

We denote by $C_y{}^h$, $2m-n$ mutually orthogonal unit normals to M^n, where here and in the sequel, indices $x, y, z, \cdots$ run over the range $\{n+1, n+2, \cdots, 2m\}$. Then the equations of Gauss are written as

(1.5) $$\nabla_c B_b{}^h=h_{cb}{}^x C_x{}^h,$$

where ∇_c denotes the operator of van der Waerden-Bortolotti covariant differentiation along M^n and $h_{cb}{}^x$ are second fundamental tensor of M^n with respect to the normals $C_x{}^h$ and those of Weingarten as

(1.6) $$\nabla_c C_y{}^h=-h_c{}^a{}_y B_a{}^h,$$

where $$h_c{}^a{}_y=h_{by}g^{ba}=h_{ycb}{}^z g^{ba}g_{zy},$$

g^{ba} denoting covariant components of the metric tensor g_{cb} of M^n, and g_{zy} the metric tensor of the normal bundle.

If the transform by F of any vector tangent to M^n is always normal to M^n, that is, if there exists a tensor field $f_b{}^x$ of mixed type such that

(1.7) $$F_i{}^hB_b{}^i=-f_b{}^xC_x{}^h.$$

We say that M^n is anti-invariant (or totally real) in M^{2m}.

For the transform by F of normal vectors $C_y{}^h$, we have equations of the form

(1.8) $$F_i{}^hC_y{}^i=f_y{}^a B_a{}^h+f_y{}^x C_x{}^h,$$

where $f_y{}^a=f_b{}^z g^{ba}g_{zy}$, which can also be written as $f_{ya}=f_{ay}$,
where $f_{ya}=f_y{}^b g_{ba}$ and $f_{ay}=f_a{}^z g_{zy}$.

From (1.7) and (1.8) we find

(1.9) $$f_y{}^b f_y{}^a=\delta_b{}^a,$$

(1.10) $$f_y{}^b f_y{}^x=0,$$

(1.11) $$f_y{}^z f_z{}^a=0,$$

(1.12) $$f_y{}^z f_z{}^x=-\delta_y{}^x+f_y{}^a f_a{}^x.$$

If $m=n$, from (1.9) we have $f_y{}^a f_a{}^x=\delta_y{}^x$ and consequently from (1.12) we find $f_y{}^z f_z{}^x=0$, that is, f_z, $f^{zy}=0$, $f_{zy}=f_z{}^x g_{xy}$ and $f^{zy}=f_x{}^y g^{xz}$ being skew-symmetric. Thus we have

(1.13) $$f_y{}^x=0.$$

In this case, equations (1.9)~(1.12) reduce to

(1.14) $$f_b{}^y f_y{}^a=\delta_b{}^a,\quad f_b{}^x f_y{}^b=\delta_y{}^x.$$

Equation of Gauss and Codazzi are respectively

(1.15) $$K_{dcb}{}^a=K_{kji}{}^h B_d{}^k{}_c{}^j{}_b{}^i{}^a{}_h+h_d{}^a{}_x h_{cb}{}^x-h_c{}^a{}_x h_{bd}{}^x,$$

(1.16) $$0=K_{kji}{}^h B_d{}^k{}_c{}^j{}_b{}^i C^x{}_h-(\nabla_d h_{cb}{}^x-\nabla_c h_{db}{}^x),$$

$K_{dcb}{}^a$ and $K_{kji}{}^h$ being curvature tensor of M^n and M^{2m} respectively,
where $B_d{}^k{}_c{}^j{}_b{}^i{}^a{}_h=B_d{}^k B_c{}^j B_b{}^i B^a{}_h$, $B_d{}^k{}_c{}^j{}_b{}^i=B_d{}^k B_c{}^j B_b{}^i$.

2. Infinitesimal variation of anti-invariant submanfold [3]

We consider an infinitesimal variation of anti-invariant submanifold M^n of a Kaehlerian manifold M^{2m} given by

(2.1) $$\bar{x}^h=x^h(y)+\xi^h(y)\varepsilon,$$

where $\xi^h(y)$ is a vetor field of M^{2m} defined along M^n and ε is an infinitesimal. We then have

(2.2) $$\bar{B}_b{}^h = B_b{}^h + (\partial_b \xi^h)\varepsilon,$$

where $\bar{B}_b{}^h = \partial_b \bar{x}^h$ are n lineary independent vectors tangent to the varied submanifold. We displace $\bar{B}_b{}^h$ parallely from the varied point $(\bar{x}^h)$ to the original point (x^h).
We then obtain the vectors

$$\tilde{B}_b{}^h = \bar{B}_b{}^h + \Gamma_{ji}{}^h(x+\xi\varepsilon)\xi^j \bar{B}_b{}^i \varepsilon$$

at the point (x^h), or

(2.3) $$\tilde{B}_b{}^h = B_b{}^h + (\nabla_b \xi^h)\varepsilon$$

neglecting the terms of order higher than one with respect to ε, where

(2.4) $$\nabla_b \xi^h = \partial_b \xi^h + \Gamma_{ji}{}^h B_b{}^j \xi^i.$$

In the sequel we always neglect term of order higher than one with respect to ε. Thus putting

(2.5) $$\delta B_b{}^h = \tilde{B}_b{}^h - B_b{}^h,$$

we have from (2.3)

(2.6) $$\delta B_b{}^h = (\nabla_b \xi^h)\varepsilon.$$

Putting

(2.7) $$\xi^h = \xi^a B_a{}^h + \xi^x C_x{}^h,$$

we have

(2.8) $$\nabla_b \xi^h = (\nabla_a \xi^h - h_b{}^a{}_x \xi^x) B_a{}^h + (\nabla_b \xi^x + h_{ba}{}^x \xi^a) C_x{}^h.$$

We now denote by $\bar{C}_y{}^h$ 2m−n mutually orthogonal unit normals to varied submanifold and by $\tilde{C}_x{}^h$ the vectors obtained from $\bar{C}_y{}^h$ by parallel displacement of $\bar{C}_y{}^h$ from the point $(\bar{x}^h)$ to (x^h). Then we have

$$\tilde{C}_y{}^h = \bar{C} + \Gamma_{ji}{}^h(x+\xi\varepsilon)\ \xi^j \bar{C}_y{}^i \varepsilon.$$

We put

(2.9) $$\delta C_y{}^h = \tilde{C}_y{}^h - C_y{}^h$$

and assume that $\delta C_y{}^h$ is of the form

(2.10) $$\delta C_y{}^h = (\eta_y{}^a B_a{}^h + \eta_y{}^x C_x{}^h)\varepsilon.$$

Then, from (2.6), (2.9) and (2.10), we have

$$\bar{C}_y{}^h = C_y{}^h - \Gamma_{ji}{}^h \xi^j C_y{}^i \varepsilon + (\eta_y{}^a B_a{}^h + \eta_y{}^x C_x{}^h)\varepsilon.$$

An infinitesimal variation given by (2.1) is called an *anti-invariant* variation if it carries an anti-invariant submanifold into an anti-invariant submanifold. For an infinitesimal variation given by (2.1), when $\xi^a = 0$, that is, when the variation vector ξ^h is normal to the submanifold we say that variation is *normal*.

Yano and Kon[3] have proved in their paper:

Theorem 2.1. *In order for an infinitesimal variation to carry an anti-invariant submanifold into an anti-invariant submanifold, it is necessary and sufficient that the variation vector ξ^h satisfies*

(2.11) $$(\nabla \xi^x) f_a{}^x = f_b{}^x (\nabla^a \xi_x).$$

In this case, the variation of $f_b{}^x$ is given by

(2.12) $$\delta f_b{}^x = [(\nabla_b \xi^a - h_b{}^a{}_y \xi^y) f_a{}^b - (\nabla_b \xi^y + h_{ba}{}^y \xi^a) f_y{}^x - f_b{}^y \eta_y{}^x]\varepsilon.$$

Lemma 2.1. (1) *In order for an infinitesimal variation to be parallel, it is necessary and sufficient that*

(2.13) $$\nabla_b \xi^x + h_{ba}{}^x \xi^a = 0.$$

(2) *A parallel variation is an anti-invariant variation.*

Applying the operator δ to (1.7) and using (2.6), (2.8) and $\delta g_{ji}=0$, we find

(2.14) $$\delta g_{cb} = (\nabla_c \xi_b + \nabla_b \xi_c - 2h_{cbx}\xi^x)\varepsilon,$$

from which

(2.15) $$\delta g^{ba} = -(\nabla^b \xi^a + \nabla^a \xi^b - 2h^{ba}{}_x \xi^x)\varepsilon.$$

A variation of a submanifold for which $\delta g_{cb}=0$ is said to be *isometric.*

Theorem 2.2. [3] *If an anti-invariant variation preserves* $f_b{}^x$ *then the variation is isometric.*

It is well known that infinitesimal variation of the second fundamental tensors is given by

(2.16) $$\delta h_{cb}{}^x = [\xi^d \nabla_d h_{cb}{}^x + h_{eb}{}^x(\nabla_b \xi^e) + h_{ce}{}^x(\nabla_b \xi^e) - h_{cb}{}^y \eta_y{}^x]\varepsilon$$
$$+ [\nabla_c \nabla_b \xi^x + K_{kji}{}^h B_c{}^j{}_b{}^i C^x{}_h C_y{}^k \xi^y - h_{ce}{}^x h_b{}^e{}_y \xi^y]\varepsilon.$$

A variation carries a minimal submanifold into a minimal submanifold, that is, $\delta(g^{cb}h_{cb}{}^x)=0$ and $h_b{}^b{}_x=0$. Then we say that the variation is *minimal.*

Pak and Ki have proved the following:

Lemma 2.2. [5] *Let* M^n *be a compact orientable submanifold of a Riemannian manifold. If an infinitesimal variation* $\bar{x}^h = x^h + \xi^h \varepsilon$ *of* M^n *is isometric and minimal, then we have*

(2.17) $$\nabla^c U_c = -K_{kjih}\xi^k B^{ji}\xi^h + \| \nabla_c \xi_x + h_{cex}\xi^e \|^2 + \| \nabla_c \xi_b - h_{cbx}\xi^x \|^2,$$

where

$$U_c = \frac{1}{2}\nabla_c(\xi_a \xi^a + \xi_x \xi^x).$$

Moreover, if $K_{kjih}\xi^k B^{ji}\xi^h \leq 0$, *then the variation is parallel.*

3. Anti-invariant minimal variation with vanishing Bochner curvature tensor

We now suppose that the ambient Kaehlerian manifold has vanishing Bochner curvature tensor [3], that is,

(3.1) $$K_{kjih} = -[g_{kh}L_{ji} - g_{jh}L_{ki} + L_{kh}g_{ji} - L_{jh}g_{ki} + F_{kh}M_{ji}$$
$$- F_{jh}M_{ki} + M_{kh}F_{ji} - M_{jh}F_{ki} - 2(F_{kj}M_{ih} + M_{kj}F_{ih})],$$

where $L_{ji} = -\dfrac{1}{2(m+2)}K_{ji} + \dfrac{1}{8(m+1)(m+2)}Kg_{ji}$, $M_{ji} = -L_{jt}F_i{}^t$,

K_{ji} and K being the Ricci tensor and scalar curvature of M^{2m} respectively.

Suppose also that a submanifold M^m of M^{2m} is anti-invariant. Then we have

(3.2) $$-K_{kjih}\xi^k B^{ji}\xi^h = [g_{kh}L_{ji} - g_{jh}L_{ki} + L_{kh}g_{ji}$$

$$-L_{jh}g_{ki}+F_{kh}M_{ji}-F_{jh}M_{ki}+M_{kh}F_{ji}$$
$$-M_{jh}F_{ki}-2(F_{kj}M_{ih}+M_{kj}F_{ih})]\xi^k B^{ji}\xi^h,$$

where $B^{ji}=B_c{}^j B_b{}^i g^{cb}$.

Substituting(2.7) into (3.2), we find

(3.3)
$$-K_{kjih}\xi^k B^{ji}\xi^h=[g_{kh}L_{ji}B^{ji}-g_{jh}L_{ki}B^{ji}+L_{kh}g_{ji}B^{ji}-L_{jh}g_{ki}B^{ji}$$
$$+F_{kh}M_{ji}B^{ji}-F_{jh}M_{ki}B^{ji}+M_{kh}F_{ji}B^{ji}-M_{jh}F_{ki}B^{ji}$$
$$-2(F_{kj}M_{ih}B^{ji}+M_{kj}F_{ih}B^{ji})](\xi^a B_a{}^k+\xi^x C_x{}^k)(\xi^b B_b{}^h$$
$$+\xi^y C_y{}^h).$$

On the ohter hand, we can easily verify that

(3.4) $M_{ji}\ B^{ji}=0,\ \ F_{ji}\ B^{ji}=0$

and

(3.5)
$$(\xi^a B_a{}^k+\xi^x C_x{}^k)(\xi^b B_b{}^h+\xi^y C_y{}^h)=\xi^a\xi^b B_a{}^k B_b{}^h+2\xi^a\xi^y B_a{}^k C_y{}^h+\xi^x\xi^y C_x{}^k C_y{}^h.$$

Substituting (3.4) and (3.5) into (3.3), we have

(3.6)
$$-K_{kjih}\xi^k B^{ji}\xi^h=[g_{kh}L_{ji}B^{ji}-g_{jh}L_{ki}B^{ji}+L_{kh}g_{ji}B^{ji}$$
$$-L_{jh}g_{ki}B^{ji}-F_{jh}M_{ki}B^{ji}-M_{jh}F_{ki}B^{ji}$$
$$-2(F_{kj}M_{ih}B^{ji}+M_{kj}F_{ih}B^{ji})](\xi^a\xi^b B_a{}^k B_b{}^h$$
$$+2\xi^a\xi^y B_a{}^k C_y{}^h+\xi^x\xi^y C_x{}^k C_y{}^h).$$

Denoting $L_{yx}=L_{ji}C_y{}^j C_x{}^i,\ \ L_{cx}=L_{ji}B_c{}^j C_x{}^i,\ \ L=L_{ji}B^{ji},\ \ L_{cb}=L_{ji}B_c{}^j{}_b{}^i.$, we have

(3.7) $L_{cb}f_y{}^c f_x{}^b=L_{ji}B_c{}^j B_b{}^i f_y{}^c f_x{}^b=L_{ji}\ F_t{}^j C_y{}^t F_s{}^i C_x{}^s=L_{yx}$

because of $L_{ji}F_t{}^j F_s{}^i=L_{ts}$ and (1.7), (1.8) and (1.13).

Using (1.7), (1.8), (1.13) and (3.7), (3.6) reduces to

(3.8)
$$-K_{kjih}\xi^k B^{ji}\xi^h=L\xi_a\xi^a+L\xi_x\xi^x+(m-2)L_{cb}\xi^c\xi^b$$
$$+2(m+2)L_{cx}\xi^c\xi^x+(m+6)L_{yx}\xi^y\xi^x$$

According to Lemma 2.2, we have

Theorem 3.1. *Suppose that* M^{2m} *is a Kaehlerian manifold with vanishing Bochner tensor and* M^m *is a compact orientable anti-invariant submanifold of* M^{2m}. *If an anti-invariant minimal variation of* M^m *preserves* $f_b{}^x$ *and*

$$[Lg_{cb}+(m-2)L_{cb}]\xi^c\xi^b+[Lg_{yx}+(m+6)L_{yx}]\xi^y\xi^x\ +2(m+2)L_{cx}\xi^c\xi^x\ \geqq\ 0,$$

then the variation is parallel and ξ^a *is harmonic vector field.*

If the variation is normal, we have

Corollary 3.2. [3] *Suppose that* M^{2m} *is a Kaehlerian manifold with vanishing Bochner curvature tensor and that* M^m *is a compact orientable anti-invariant submanifold of* M^m. *If an anti-invariant normal variation of* M^m *preserves* $f_b{}^x$ *and the mean curvature vector and*

$$[Lg_{yx}+(m+6)L_{yx}]\xi^y\xi^x\ \geqq\ 0,$$

then the variation is parallel.

REFERENCES

[1]. B.Y. Chen and K. Yano, On the theory of normal variations, to appear to *J. Differential Geometry.*

[2]. K. Yano, Infinitesimal variation of submanifolds, to appear in *Kodai Math.* Sem. Rep.

[3]. K. Yano and M. Kon(1976), Infinitesimal variations of anti-invariant submanifolds of Kaehlerian manifolds, *Kyungpook Math. J.* **16**, 33~47.

[4]. ——, *Anti-invariant submanifolds,* Marcel Dekker, 1976.

[5]. Eulyong Pak and U-Hang Ki(1977), On infinitesimal minimal variation of anti-invariant submanifolds of a Kaehlerian manifold, *J. Korean Math. Soc.* **14**, 71~80.