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A Note On Injectivity Of Unital Modules

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1. Introduction.

In this paper we investigate for injectivity of a unital module M_R by using the Baer's injectivity condition; "M_R is u-injective if and only if for any right ideal I of R and each $f\in Hom_R(I, M)$ there exists $m\in M$ such that $f(x)=mx$ for any $x\in I$". We define the following.

(1) S is a nonempty subset of a ring R and M_R is a right R-module

$$_RS=\{r\in R\mid Sr=0\}$$
$$_MS=\{x\in M\mid xS=0\}$$

(2) If R is a ring with unit 1 and for any $x\in M$, $x1=x$, then M_R is a unital module.

(3) A unital module M_R is a u-injective module in case each unital module B_R has the following properties that if A_R is a submodule of B_R, then any $f\in Hom_R(A, M)$ can be induced by an element $g\in Hom_R(B, M)$.

(4) I is a right ideal of a ring R, unital module M_R is I-complete if for any $f\in Hom_R(I, M)$ there is an element $m\in M$ such that $f(x)=mx$ for every $x\in I$.

2. Injectivity of a unital module

Theorem 1. *Let M_R be a unital module. For every $x\in R$, $_M(_Rx)\subset Mx$ if and only if M_R is xR-complete.*

Proof. (Necessity) We assume $_M(_Rx)\subset Mx$. If $f\in Hom_R(xR, M)$, then for any $r\in {_Rx}$

$$f(x)r=f(xr)=f(0)=0$$

Hence $f(x)\in {_M(_Rx)}\subset Mx$. And so there is a $m\in M$ such that $f(x)=mx$. For every $y=xr\in xR$

$$f(y)=f(xr)=f(x)r=(mx)r=m(xr)=my$$

Therefore M_R is xR-complete.

(Sufficiency). Let $y\in {_M(_Rx)}$ and we define $f:xR\to M$ as $f(xr)=yr$. If $xr_1=xr_2$, then $x(r_1-r_2)=0$ and so $r_1-r_2\in {_Rx}$. Therefore $y(r_1-r_2)=0$, that is,

$$f(xr_1)=f(xr_2).$$

Hence $f:xR\to M$ is a mapping. Next for any xr_1, $xr_2\in xR$, $r\in R$

$$f(xr_1+xr_2)=f(x(r_1+r_2))=y(r_1+r_2)=yr_1+yr_2=f(xr_1)+f(xr_2).$$

$$f((xr_1)r)=f(x(r_1r))=y(r_1r)=(yr_1)r=f(xr_1)r$$

This proves $f\in Hom_R(xR, M)$. And for any $xr\in xR$, there is $m\in M$ such that $f(xr)=m(xr)$. In particular $y=y1=f(x1)=m(x1)=mx\in Mx$. That is, ${}_M({}_Rx)\subseteq Mx$.

Corollary. *M_R is a unital module and ${}_Rx=0$ for $x\in R$, $x\neq 0$, $Mx=M$ if and onlt if M_R is xR-complete.*

Proof. (Necessity) ${}_M({}_Rx)\subset M=Mx$

For any $f\in Hom_R(xR, M)$, there exists $m\in M$ such that $f(y)=my$ for any $y\in xR$ by Theorem 1. Therefore M_R is xR-complete.

(Sufficiency) ${}_Rx=\{x\in R\mid xr=0\}=\{0\}$

And so ${}_M({}_Rx)=M$

And by Theorem 1, ${}_M({}_Rx)\subset Mx$. That is $M\subset Mx$. Therefore $M=Mx$.

Theorem 2. *M_R is a unital module. Then M_R is u-injective if and only if M_R is an injective unital module.*

Proof. (Necessity) For any R-module B_R we define $B^1=B1$ and $B^0=\{x\in B\mid x1=0\}$. Then B^1 and B^0 are submodules of B, since for any $x1$, $y1\in B^1$, $r\in R$,

$$x1-y1=(x-y)1,\ (x1)r=x(1r)=x(r1)=(xr)1\in B^1$$

And for any $x, y\in B^0$, $r\in R$

$$(x-y)1=x1-y1=0-0=0$$

hence $x-y\in B^0$. And

$$(xr)1=x(r1)=x(1r)=(x1)r=0r=0$$

hence $xr\in B^0$.

For any $x\in B^1\cap B^0$, there is a $y\in B$ such that $x=y1$ and $x1=0$, therefore $x=y1=(y1)1=x1=0$ so that $B^1\cap B^0=0$. For any $x\in B$, if $x1\neq 0$, then

$$(x-x1)1=x1-x1=0 \text{ and so } x-x1\in B^0$$

And $x=x1+(x-x1)\in B^1\oplus B^0$

If $x1=0$, then $x\in B^0$ and so $x=0+x\in B^1+B^0$. Therefore $B=B^1\oplus B^0$. And B^1 is a unital module because for any $x\in B^1$, there is $y\in B$ such that $x=y1$ and $x=y1=(y1)1=x1$. Let A_R be any submodule of B_R. Then there exist submodules of A such that $A=A^1\oplus A^0$ where A^1 is a unital submodule of B^0. Let $f\in Hom_R(A, M)$, and define $f'\in Hom_R(A^1, M)$ as $f'(a1)=f(a1)$. Then there is $g'\in Hom_R(B^1, M)$ such that $g'\mid A^1=f'$(by hypothesis) Now we define a mapping

$$g:B\rightarrow M \text{ as } g(x_1+x_0)=g'(x_1)$$

Then for any $a=a_1+a_0\in A$

$$f(a_0)=f(a_0)1=f(a_01)=f(0)=0$$

Hence $g(a_1+a_0)=g'(a_1)=f'(a_1)=f(a_1)=f(a_1)+0=f(a_1)+f(a_0)=f(a_1+a_0)$. And for any x, $y\in B$, $r\in R$,

$$g(x+y)=g((x_1+x_0)+(y_1+y_0))=g((x_1+y_1)+(x_0+y_0))$$
$$=g'(x_1+y_1)=g'(x_1)+g'(y_1)=g(x_1+x_0)+g(y_1+y_0)$$

$$=g(x)+g(y).$$
$$g(xr)=g((x_1+x_0)r)=g(x_1r+x_0r)=g'(x_1r)=g'(x_1)r=g(x_1+x_0)r$$
$$=g(x)r$$

So that $g \in Hom_R(B, M)$ and $g|A=f$.

(Sufficiency) By definition of u-injectivity if M_R is unital injective, then M_R is u-injective.

Corollary. Let *V be a finite dimensional vector space over a field K. Then V_K is injective unital.*

Proof. Let A_K be a submodule of any unital module B_K. And let $\{a_1, a_2, \cdots, a_r\}$ be a basis of A_K. Since $\{a_i\}$ is linearly independent, it can be extended to a basis of B, say $\{a_1, \cdots, a_r, c_1, \cdots, c_s\}$, Let C be the submodule of B genetrated by $\{c_1, \cdots, c_s\}$. Since $\{a_i, c_i\}$ generates B, $B=A+C$. On the other hand, $A \cap C=\{0\}$. Accordingly, $B=A \oplus C$. For any $f \in Hom_K(A, V)$, we define $g: B \to V$ as $g(a+c)=f(a)$. Then for any a_1+c_1, $a_2+c_2 \in B$, $h \in K$ if $a_1+c_1=a_2+c_2$, then $a_1-a_2=c_2-c_1$. And since $A \cap C=\{0\}$, $a_1-a_2=c_2-c_1=0$. Hence $a_1=a_2$, $f(a_1)=f(a_2)$. Therefore,

$$g(a_1+c_1)=g(a_2+c_2)$$

And
$$g((a_1+c_1)+(a_2+c_2))=g((a_1+a_2)+(c_1+c_2))$$
$$=f(a_1+a_2)$$
$$=f(a_1)+f(a_2)=g(a_1+c_1)+g(a_2+c_2),$$
$$g((a_1+c_1)h)=g(a_1h+c_1h)=f(a_1h)=f(a_1)h=g(a_1+c_1)h$$

Accordingly $g \in Hom_K(B, V)$ with $g|A=f$. Hence V_K is u-injective, by Theorem 2, it is injective.

3. Some results

Let R be an integral domain with unit and M_R is unital. Then M_R is divisible if and only if M_R is xR-complete. And if R is a principal right ideal ring and integral domain, then by Baer's injectivity condition we know that M_R is u-injective if and only if M_R is xR-complete. By Theorem 2, if R is a principal right ideal ring and integral domain, then M_R is injective unital module if and only if M_R is xR-complete. Consequently we know that if R is a principal right ideal ring and integral domain, then M_R is injective unital module if and only if M_R is divisible. For example, a Z-module G is injective if and only if it is divisible. So the concepts "injective" and "divisible" are equivalent for abelian groups. The group Q, consisting of the rational numbers modulo integer, is a Z-injective module.

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