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Completion Of The Space Of Distribution Under Paul Levy Metrics

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We consider the space of all distribution functions and we give the Paul Levy metrics $L(F, G)$, between two distribution function $F(x)$ and $G(x)$, as the infimum of all positive h such that the inequality

(1) $$F(x-h)-h \leq G(x) \leq F(x+h)+h$$

holds for all x.

Lemma. *The sequence of distribution functions $\{F_n(x)\}$ converges weakly to the distribution function $F(x)$ if, and only if, $\lim_{n\to\infty} L(F_n, F)=0$.*

Proof. First of all, we prove that the condition is necessary. Since $F(x)$ is a distribution function, It is always possible to find two continuity points a and b of $F(x)$ such that

(2) $$F(a)\leq \epsilon/2, \quad 1-F(b)\leq \epsilon/2$$

for an arbitrary positive number ϵ.

We select m continuity points of $F(x)$ in the interval $[a, b]$ such that

$$a=t_0<t_1<t_2<\cdots<t_m<t_{m+1}=b$$
$$t_{k+1}-t_k<\epsilon \quad (k=0, 1, 2, \cdots, m).$$

Since the t_k are continuity points of $F(x)$, we conclude from assumption that it is possible to choose a number N so large that

(3) $$|F_n(t_k)-F(t_k)|\leq \epsilon/2$$

for $n\geq N$ and $k=0, 1, 2, \cdots, m+1$.

In order to prove that $\lim L(F_n, F)=0$, we have to show that for arbitrary $\epsilon>0$ the relation

(4) $$F(x-\epsilon)-\epsilon\leq F_n(x)\leq F(x+\epsilon)+\epsilon$$

holds for all x, provided n is chosen sufficiently large. We have to distinguish three cases:

Case I. If $x\leq a$, then we conclude from (2) and (3) that

$$F_n(x)\leq F_n(a)\leq F(a)+\epsilon/2\leq \epsilon\leq F(x+\epsilon)+\epsilon,$$
$$F_n(x)\geq 0\geq F(a)-\epsilon/2\geq F(x-\epsilon)-\epsilon$$

provided $n\geq N$.

Case II. If $t_{k-1} \leq x \leq t_k$, then we conclude from (3) that

$$F_n(x) \leq F_n(t_k) \leq F(t_k) + \epsilon/2 \leq F(x+\epsilon) + \epsilon,$$
$$F_n(x) \geq F_n(t_{k-1}) - \epsilon/2 \geq F(x-\epsilon) - \epsilon$$

provided $n \geq N$.

Case III. If $x \geq b$, then we conclude from (2) and (3) that

$$F_n(x) \leq 1 \leq F(b) + \epsilon/2 \leq F(x+\epsilon) + \epsilon,$$
$$F_n(x) \geq F_n(b) \geq F(b) - \epsilon/2 \geq 1 - \epsilon \geq F(x-\epsilon) - \epsilon$$

provided $n \geq N$. Therefore (4) holds for all x if $n \geq N$. Since ϵ is arbitrary, this means that $\lim_{n\to\infty} L(F_n, F) = 0$.

To prove the sufficiency of the condition, let x_o be a continuity point of $F(x)$, and let ϵ be an arbitrary possitive number; then there exists a $\delta > o$ such that

(5) $$|F(x) - F(x_o)| < \epsilon$$

for all x for which

$$|x - x_o| < \delta.$$

Let $\gamma = \min(\epsilon, \delta)$ and choose N so large that $L(F_n, F) < \gamma$ for $n \geq N$. According to (1) and (5) we have

$$F_n(x_o) \geq F(x_o - \gamma) - \gamma \geq F(x_o) - 2\epsilon$$

and $$F_n(x_o) \leq F(x_o + \gamma) + \gamma \leq F(x_o) + 2\epsilon,$$

hence $$|F_n(x_o) - F(x_o)| \leq 2\epsilon.$$

Since ϵ and the continuity point x_o are arbitrary, this means that $\lim_{n\to\infty} F_n(x) = F(x)$.

We use the first theorem of Helly (Every sequence $\{F_n(x)\}$ of uniformly bounded functions of M contains a subsequence $\{F_{nk}(x)\}$ which converges to some function $F(x) \in M$ at every continuity point of $F(x)$. ref. to I) to prove our main theorem.

Theorem. *The space of distribution functions with the Levy distance $L(F, G)$ is complete.*

Proof. To prove the theorem we assume that for any $\epsilon > 0$ there exist an $N = N(\epsilon)$ such that

(6) $$L(F_n, F_m) \leq \epsilon$$

for $n, m \geq N$. Since the $F_n(x)$ are distribution functions, we have $0 \leq F(x) \leq 1$ according to the first theorem of Helly.

We note that it is always possible to find an x_m such that $F(x_m) \leq \epsilon$ and we select $n_k > N$, $m > N$. We see from (6) that $L(F_{nk}, F_m)$ and choose a continuity point x of $F(x)$ for which $x < x_m - \epsilon$.

then $$F_{nk}(x) \leq F_{nk}(x_m - \epsilon) \leq F_m(x_m) + \epsilon \leq 2\epsilon.$$

Therefore $F(x) = \lim_{k\to\infty} F_{nk}(x) \leq 2\epsilon$ if x is a continuity point of $F(x)$ and $x < x_m - \epsilon$. This means that $F(-\infty) = 0$.

In the same way, one can show that $F(+\infty) = 1$, so that $F(x)$ is a distribution function. It follows from lemma that

(7) $$\lim_{k\to\infty} L(F_{nk}, F) = 0.$$

We conclude finally from (6), (7), and the triangle inequality that $L(F_n, F)$ can be made arbitrary small by choosing n sufficiently large. The sequence $F_n(x)$ converges, therefore, to F(x) in the metric of the Levy distance.

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