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On The Relation Between Extensions Of Topologies And Quasi-uniformities

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1. Introduction

In this paper, we shall investigate the answers to the following questions;

i) Let Q be the some property of the quasi-uniformity. If $(X, \mathfrak{U})$ has the property Q, under what conditions will the extension of the quasi-uniformity also have the property Q?

ii) Let $(X, \mathfrak{I})$ be a topological space and $\mathfrak{U}$ be the PERVIN's quasi-uniformity by the topology $\mathfrak{I}$. Them can the topology generated by the extension quasi-uniformity of PERVIN's quasi-uniformity $\mathfrak{U}$ be represented by LEVIN's simple extension of the given topology $\mathfrak{I}$?

Let $(X, \mathfrak{U})$ be a quasi-uniform space and let A be a given subset of X. We consider the smallest quasi-uniformity which contains the quasi-uniformity $\mathfrak{U}$ and $\{A\times A\cup A'\times A'\}$, and will define this smallest quasi-uniformity which contains the quasi-uniformity as an extension of quasi-uniformity $\mathfrak{U}$.

2. Main results

Definition 2.1. Let $(X, \mathfrak{U})$ is a quasi-uniform space. Let $\mathcal{L}=\{U\cap(A\times A\cup A'\times A'):U\in\mathfrak{U}\}$ and $\mathfrak{U}^*$ be a quasi-uniformity with $\mathcal{L}$ as base. Then $\mathfrak{U}^*$ is called an ***extension*** of $\mathfrak{U}$. In this case we write $\mathfrak{U}^*=\mathfrak{U}(A)$.

For a subset A of a topological space X, A′ denotes the complement $X\sim A$ of A. $\mathfrak{I}(\mathfrak{U}(A))$ denotes the quasi-uniform topology of the quasi-uniformity $\mathfrak{U}(A)$.

Let$(X, \mathfrak{I})$ be a topological space and let $\mathfrak{U}$ be PERVIN's quasi-uniformity by the topology $\mathfrak{I}$. $\mathfrak{U}(A)$ is the extension of $\mathfrak{U}$ and $\mathfrak{I}$ $(\mathfrak{U}(A))$ is the quasi-uniform topology of the quasi-uniformity $\mathfrak{U}(A)$.

Lemma 2.2 (a) *A is cl-open (closed and open) in $(X, \mathfrak{I}(\mathfrak{U}(A))$*

(b) $\{O\cap A,\ O\cap A':O\in\mathfrak{I}\}$ *is a base for* $\mathfrak{I}(\mathfrak{U}(A))$.

Proof. (a) is obvious.

(b) Let $O\in\mathfrak{I}(\mathfrak{U}(A))$. Then for every $x\in O$, there exists U in $\mathfrak{U}$ such that $(U\cap(A\times A\cup A'\times A'))[x]\subset O$. If $x\in A$, then $U[x]\cap A\subset O$. Hence there exists O_x in $\mathfrak{I}$ such that $x\in O_x$

and $O_x \cap A \subset U[x] \cap A \subset O_x$. If $x \in A'$, then $U[x] \cap A' \subset O$. Hence there exists O_x in $\mathfrak{J}$ such that $x \in O_x$ and $O_x \cap A' \subset U[x] \cap A' \subset O$. Hence $\{O \cap A,\ O \cap A' : O \in \mathfrak{J}\}$ is a base for $\mathfrak{J}(\mathfrak{U}(A))$.

Theorem 2.3. $\mathfrak{J}(\mathfrak{U}(A)) = \mathfrak{J}(A)(A')$.

Proof. By lemma 2.2, $\mathfrak{J}(\mathfrak{U}(A)) \subset \mathfrak{J}(A)(A')$.

Let $O^* = O_1 \cup (O_2 \cap A)$ and $O^{**} = O_3 \cup (O_4 \cap A)$ for $O_1, O_2, O_3, O_4, \in \mathfrak{J}$.

Then $O^* \cup (O^{**} \cap A') = O_1 \cup (O_2 \cap A) \cup (O_3 \cup A')$.

Therefore by lemma 2.2, $O^* \cup (O^{**} \cap A') \in \mathfrak{J}\ (\mathfrak{U}(A))$.

Hence $\mathfrak{J}(\mathfrak{U}(A)) \supset \mathfrak{J}(A)(A')$. Thus $\mathfrak{J}(\mathfrak{U}(A)) = \mathfrak{J}\ (A)(A')$.

We obtain the following lemma from lemma 2.2.

Lemma 2.4. $(A,\ \mathfrak{J} \cap A) = (A,\ \mathfrak{J}(\mathfrak{U}(A)) \cap A)$.
$(A', \mathfrak{J} \cap A') = (A', \mathfrak{J}(\mathfrak{U}(A)) \cap A')$.

Theorem 2.5. *Let A and A′ be the subspaces of the space* $(X, \mathfrak{J})$ *and let* $\mathfrak{J}^*$ *be the weak topology determined by* $\{A, A'\}$. *Then* $\mathfrak{J}^* = \mathfrak{J}(\mathfrak{U}(A))$.

Proof. By lemma 2.4, $\mathfrak{J}(\mathfrak{U}(A)) \subset \mathfrak{J}^*$. Let U be a member of $\mathfrak{J}^*$. Then since $U \cap A$ and $U \cap A'$ are open in$(A,\ \mathfrak{J} \cap A)$ and $(A', \mathfrak{J} \cap A')$ respectively, $U \cap A = \bigcup_\alpha (O_\alpha \cap A)$ and $U \cap A' = \bigcup_\beta (O_\beta \cap A')$ for O_α, $O_\beta \in \mathfrak{J}$. Since $U = (U \cap A) \cup (U \cap A') = (\bigcup_\alpha (O_\alpha \cap A)) \cup (\bigcup_\beta (O_\beta \cap A'))$, U is a member of $\mathfrak{J}\ (\mathfrak{U}(A))$ bylemma 2.2.

We obtain the following corollary from lemma 2.4 and theorem 2.5.

Corollary 2.6. $(X, \mathfrak{J}(\mathfrak{U}(A)))$ *has the property P iff* $(A, \mathfrak{J} \cap A)$ *and* $(A', \mathfrak{J} \cap A')$ *have the property P, where P is* $T_i (i=0, 1, 2)$, *regular, completely regular, normal, compact, countably compact, Lindelöf and separable.*

Lemma 2.7. *In definition* 2.1, $\mathfrak{L}$ *is a base for some quasi-uniformity.*

Proof. (a) Since $\Delta \subset U$ and $\Delta \subset A \times A \cup A' \times A'$ for every $U \in \mathfrak{U}$, $\Delta \subset \tilde{U}$ for every $\tilde{U} \in \mathfrak{L}$.

(b) Let $\tilde{U} = U \cap (A \times A \cup A' \times A')$. Then since $U \in \mathfrak{U}$, there exists V in $\mathfrak{U}$ such that $V \circ V \subset U$. Let $\tilde{V} = V \cap (A \times A \cup A' \times A')$, then $\tilde{V} \circ \tilde{V} \subset \tilde{U}$.

(c) For $\tilde{U}, \tilde{V} \in \mathfrak{L}$, let $\tilde{U} = U \cap (A \times A \cup A' \times A')$ and $\tilde{V} = V \cap (A \times A \cup A' \times A')$. Then $\tilde{U} \cap \tilde{V} = (U \cap V) \cap (A \times A \cup A' \times A')$. Since $U \cap V \in \mathfrak{U}$, $\tilde{U} \cap V \in \mathfrak{L}$.

Theorem 2.8. *If* $(X, \mathfrak{U})$ *is pseudo metrizable, then* $(X, \mathfrak{U}(A))$ *is pseudo metrizable.*

Proof. Let $\mathcal{A} = \{U_i\}$ be a countable base for the quasi-uniformity $\mathfrak{U}$. Then $\{U_i \cap (A \times A \cup A' \times A') : U_i \in \mathcal{A}\}$ is a countable base for the quasi-uniformity $\mathfrak{U}(A)$.

Theorem 2.9. $(X, \mathfrak{U}(A))$ *is precompact iff* $(X, \mathfrak{U})$ *is precompact such that for each* U in $\mathfrak{U}$ there exists a finite subset $F^U$of X such that $U[F^U] = X$, $F^U \cap A \neq \phi$ *and* $F^U \cap A' \neq \phi$.

Proof. Suppose(X, $\mathfrak{U}(A)$) is precompact. Let $U \in \mathfrak{U}$. Then $U \cap (A \times A \cup A' \times A') \subset U$. Since $(X, \mathfrak{U}(A))$ is precompact, there exists a finite subset F of X such that

$(U\cap(A\times A\cup A'\times A'))[F]=U[F]\cap((A\times A)[F]\cup(A'\times A')[F])=X.$
Thus $U[F]=X$. Here $F\cap A\neq\phi$, $F\cap A'\neq\phi$.
If $F\cap A=\phi$, since $(A\times A)[F]=\phi$, $(U\cap(A\times A\cup A'\times A'))[F]=U[F]\cap A'\neq X$. If $F\cap A'=\phi$, since $(A'\times A')[F]=\phi$, $(U\cap(A\times A\cup A'\times A'))[F]=U[F]\cup A\neq X$.
Suppose $(X,\mathfrak{U})$ is precompact such that for each U in $\mathfrak{U}$ there exists a finite subset F^U of X such that $U[F^U]=X$, $F^U\cap A\neq\phi$ and $F^U\cap A'\neq\phi$. Let $\tilde{U}\in\mathfrak{U}(A)$.
Then there exists a member U of $\mathfrak{U}$ such that $U\cap(A\times A\cup A'\times A')\subset\tilde{U}$.
Since $U\in\mathfrak{U}$, there exists a finite subset F^U of X such that $U[F^U]=X$, $F^U\cap A\neq\phi$ and $F^U\cap A'\neq\phi$. Since $(U\cap(A\times A\cup A'\times A'))[F^U]=U[F^U]\cap((A\times A)[F^U]\cup(A'\times A')[F^U])=X\cap(A\cup A')=X$ and $(U\cap(A\times A\cup A'\times A'))[F^U]\subset\tilde{U}[F^U]$, $\tilde{U}[F^U]=X$.

Theorem 2.10. *Let $(X,\mathfrak{U})$ be a complete uniform space and let $(X,\mathfrak{T})$ be a T_2-space. $\mathfrak{U}(A)$ is the extension of the quasi-uniformity $\mathfrak{U}$. Where $\mathfrak{J}$ is the topology of the quasi-uniformity $\mathfrak{U}$. Then (a) If $A\in\mathfrak{J}$, then $(X,\mathfrak{U}(A))$ is complete iff if Cauchy filter $\mathfrak{F}$ in $(X,\mathfrak{U}(A))$ converges to a point of A' with respect to $\mathfrak{J}$, then $A'\in\mathfrak{F}$.*
(b) If $A'\in\mathfrak{J}$, then $(X,\mathfrak{U}(A))$ is complete iff if Cauchy filter $\mathfrak{F}$ in $(X,\mathfrak{U}(A))$ converges to a point of A with respect to $\mathfrak{J}$, then $A\in\mathfrak{F}$.
(c) If A is not open and not closed in $(X,\mathfrak{J})$, then $(X,\mathfrak{U}(A))$ is complete iff if Cauchy filter $\mathfrak{F}$ in $(X,\mathfrak{U}(A))$ converges to a point of A with respect to $\mathfrak{J}$ then $A\in\mathfrak{F}$, and if Cauchy filter $\mathfrak{F}$ in $(X,\mathfrak{U}(A))$ converges to a point of A' with respect to $\mathfrak{J}$ then $A'\in\mathfrak{F}$.

Proof. (a) Only if: Let $\mathfrak{F}$ be Cauchy filter in $(X,\mathfrak{U}(A))$ and converge to a point x in A' with respect to $\mathfrak{J}$. Then since $(X,\mathfrak{U}(A))$ is complete, $\mathfrak{F}$ converges to a point y with respect to $\mathfrak{J}(\mathfrak{U}(A))$.
In fact, $\mathfrak{F}$ converges to y with respect to $\mathfrak{J}$ since $\mathfrak{J}\subset\mathfrak{J}(\mathfrak{U}(A))$. Since $(X,\mathfrak{J})$ is T_2-space, $x=y$. Hence $A'\in\mathfrak{F}$.
If: Let $\mathfrak{F}$ be a Cauchy filter in $(X,\mathfrak{U}(A))$. Then $\mathfrak{F}$ is a Cauchy filter in $(X,\mathfrak{U})$ since $\mathfrak{U}\subset\mathfrak{U}(A)$. Since $(X,\mathfrak{U})$ is complete, $\mathfrak{F}$ converges to a point x_0 in $(X,\mathfrak{J})$.
In fact $\mathfrak{F}$ converges to x_0 for $\mathfrak{J}(\mathfrak{U}(A))$ because if $x_0\in A'$, then $O\cap A'$ is a member of $\mathfrak{F}$ for O containing x_0 since $A'\in\mathfrak{F}$.

Lemma 2.11. $(A,\mathfrak{U}\cap A\times A)=(A,\mathfrak{U}(A)\cap A\times A)$
$(A',\mathfrak{U}\cap A'\times A')=(A',\mathfrak{U}(A)\cap A'\times A')$

Definition 2.12. A quasi-uniform space $(X,\mathfrak{U})$ is *quasi-uniformly locally compact* iff there is a member U of $\mathfrak{U}$ such that U [x] is compact for each x in X.

Theorem 2.13. *$(X,\mathfrak{U}(A))$ is quasi-uniformly locally compact iff there exists $\tilde{U}$ in $U(A)$ such that $(\tilde{U}\cap A\times A)[x]$ is compact in $(A,\mathfrak{J}\cap A)$ and $(\tilde{U}\cap A\times A)[x]$ is compact in $(A',\mathfrak{J}\cap A')$.*

Proof. Only if: Since $(X,\mathfrak{U}(A))$ is quasi-uniformly locally compact, there exists $\tilde{U}$ in $\mathfrak{U}(A)$ such that $\tilde{U}[x]$ is compact in $(X,\mathfrak{J}(\mathfrak{U}(A)))$ for every x in X. By lemma 2.11,

$\tilde{U}\cap A\times A\in\mathfrak{U}\cap A\times A$ and $\tilde{U}\cap A'\times A'\in\mathfrak{U}\cap A'\times A'$. By lemma 2.2 and lemma 2.4, $(\tilde{U}\cap A\times A)[x]=\tilde{U}[x]\cap A$ is compact in $(A,\mathfrak{J}\cap A)$ for $x\in A$ and $(\tilde{U}\cap A'\times A')[x]=U[x]\cap A'$ is compact in $(A',\mathfrak{J}\cap A')$ for $x\in A'$.

If: Let $(\tilde{U}\cap A\times A)[x]$ be compact in $(A,\mathfrak{J}\cap A)$ for $x\in A$ and $(\tilde{U}\cap A'\times A')[x]$ be compact in $(A',\mathfrak{J}\cap A')$ for $x\in A'$. Let $\tilde{U}^*=\tilde{U}\cap(A\times A\cup A'\times A')$. Then $U^*\in\mathfrak{U}(A)$, $\tilde{U}^*[x]=\tilde{U}[x]\cap A$ for $x\in A$ and $\tilde{U}^*[x]=\tilde{U}[x]\cap A'$ for $x\in A'$. Since $\tilde{U}[x]\cap A$ is compact in $(A,\mathfrak{J}\cap A)$ and $\tilde{U}[x]\cap A'$ is compact in $(A',\mathfrak{J}\cap A')$, $\tilde{U}^*[x]$ is compact in $(X,\mathcal{T}(\mathfrak{u}(A)))$ for every $x\in X$.

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