韓國數學教育學會誌〈數學敎育〉
1977.12. Vol.XVI No.1

Journal of the Korea Society of Mathematical Education
Dec. 1977. Vol.XVI No.1

A Note On The Quasi-Pseudo-Metrizability

By Jung Wan Nam

Gyeonsang National University, JinJu, Korea

1. Introduction

The notion of quasi-pseudo-metric spaces which is an extension of pseudo-metric spaces was introduced by W. A. Wilson and has been studied by M. Sion, G. Zelmer [3] and C. W. Patty [4].

A quasi-pseudo-metric for a set X is a mapping p of $X \times X$ into the non-negative reals satisfying, for all $x, y, z \in X$, the following axioms

i) $p(x, x) = 0$,

ii) $p(x, z) \leq p(x, y) + p(y, z)$.

If x is a point of a set X with a quasi-pseudo-metric p, and ε is any positive real number, then the set of all points $y \in X$ such that $p(x, y) < \varepsilon$ will be called the ball with center x and radius ε and will be denoted by $S_p(x, \varepsilon)$. If X is a set with a quasi-pseudo-metric p, the topology for X which is obtained by using, as a base, the family of all balls of points is said to be induced by the quasi-pseudo-metric p. A quasi-pseudo-metric space is a pair (X, p) such that p is a quasi-pseudo-metric for X. A topological space $(X, \mathfrak{I})$ is quasi-pseudo-metrizable iff there exists a quasi-pseudo-metric for X which induces $\mathfrak{I}$.

The purpose of this paper is to prove the following theorems.

Theorem 1. Let $(X, \mathfrak{I})$ be a topological space. If there exists a base $\mathfrak{L}$ of $\mathfrak{I}$ such that $\mathfrak{L}_x = \{B \in \mathfrak{L} \mid x \in B\}$ is a countable decreasing chain (that is, $\mathfrak{L}_x = \{B_1^x, B_2^x, \cdots\}$, $B_1^x \supsetneq B_2^x \supsetneq \cdots$) for each $x \in X$, then $(X, \mathfrak{I})$ is quasi-pseudo-metrizable.

Theorem 2. Let $(X, \mathfrak{I})$ be a first countable space, and $X_1 = \{x \in X \mid \mathfrak{I}_x$ has no minimal member with respect to $\subset\}$, where $\mathfrak{I}_x = \{O \in \mathfrak{I} \mid x \in O\}$ for each $x \in X$. If X_1 is a finite set and $\cup \{V_x \mid x \in X - X_1\} \subset X - X_1$, (where V_x is the minimal member of $\mathfrak{I}_x$), then $(X, \mathfrak{I})$ is quasi-pseudo-mtrizable.

2. Main Theorems

Theorem 1. *Let $(X, \mathfrak{I})$ be a topologial space. If there exists a base $\mathfrak{L}$ of $\mathfrak{I}$ such that $\mathfrak{L}_x = \{B \in \mathfrak{L} \mid x \in B\}$ is a countable decreasing chain (that is, $\mathfrak{L}_x = \{B_1^x, B_2^x, \cdots\}$, $B_1^x \supsetneq B_2^x \supsetneq \cdots$) for each $x \in X$, then $(X, \mathfrak{I})$ is quasi-pseudo-metrizable.*

Proof. Let $\mathfrak{L}$ be a base of $\mathfrak{I}$ satisfying the condition of the theorem. Choose a relation

$<$ which well orders the family $\mathfrak{B}$. For each $x\in X$, let $B_{x,1}$ be the first member of $\mathfrak{B}_x$ with respect to the order $<$, $B_{x,2}$ be the first member of $\{B\in\mathfrak{B}_x | B\subsetneqq B_{x,1}\}$ if this family is non-void. In this way, we obtain $\mathfrak{B}'_x=\{B_{x,1}, B_{x,2},\cdots\}$. Then, for each B^x_i, there cxists $B_{x,j}$ such that $B^x_i\supset B_{x,j}$.

For $A\subset X$, let

$$p_A(x, y)=\begin{cases}1 : x\in A,\ y\in X-A,\\ 0 : \text{otherwise.}\end{cases}$$

For each $u\in X$, define p_u as following

$$p_u(x, y)=\sum_{i=1}\frac{1}{2^i}p_{B_{u,i}}(x, y),$$

and $\qquad p(x, y)=\sup\{p_u(x, y) | u\in X\}$.

Clearly, p_u is a quasi-pseudo-metric on X for each $u\in X$ and consequently p is a quasi-pseudo-metric for X.

We assert that the topology $\mathfrak{I}_p$ induced by p equals to $\mathfrak{I}$.

Let $x\in B\in\mathfrak{B}$, then there exists $B_{x,n}\subset B$. Let $\varepsilon=\frac{1}{2^n}$.

If $p(x, y)<\varepsilon$, then $p_x(x, y)=\sum_{i=1}\frac{1}{2^i}p_{B_{x,i}}(x, y)<\varepsilon$.

Hence $y\in B_{x,n}$. It follows that $S_p(x,\varepsilon)\subset B$. Thus $\mathfrak{I}$ is coaser than $\mathfrak{I}_p$.

Let $x\in S_p(x,\varepsilon)$, $\varepsilon>0$.

Case 1. $\mathfrak{B}_x$ is a finite family, ($\mathfrak{B}_x=\{B^x_1, B^x_2, \cdots, B^x_j\}$).

For each $y\in B^x_j$, we have $p(x, y)=0$, and hence $B^x_j\subset S_p(x,\varepsilon)$.

Case 2. $\mathfrak{B}_x$ is not a finite family.

Let $\qquad U_i=\{B_{u,i} | u\in X\}$ $(\subset\mathfrak{B})$, $i=1, 2,\cdots$,

$\qquad U_i(x)=\{B\in U_i | x\in B\}$ $(\subset\mathfrak{B}_x)$.

We shall show $U_i(x)\subset\{B^x_1, B^x_2, \cdots, B^x_r=B_{x,i}\}$ for each I. $\cdots$(*)

Let $x\in B_{u,1}$, $B_{x,1}=B^x_p$, then $B_{u,1}=B^x_l$ for some l. If $l>P$, then $u\in B_{x,1}$, since $B_{u,1}\subset B_{x,1}$. This is contradict to $B_{x,1}<B^x_l=B_{u,1}$. Hence $U_1(x)\subset\{B^x_1, B^x_2, \cdots, B^x_p=B_{x,1}\}$. Assume $U_i(x)\subset\{B^x_1, B^x_2, \cdots, B_{x,i}\}$. Let $x\in B_{u,i+1}$, $B_{x,i+1}=B^x_s$. Then $B_{u,i+1}=B^x_l$ for some l. Suppose $l>s$. Then by induction hypothesis and $B_{x,1}<B_{x,2}<\cdots<B_{x,n}<B_{x,n+1}<\cdots$, we have $B_{u,i+1}\leqq B_{x,i+1}$. This is contradict to $B^x_l>B_{x,i+1}$. Therefore $U_{i+1}(x)\subset\{B^x_1, B^x_2, \cdots, B^x_s=B_{x,i+1}\}$.

Now, choose n such that $\frac{1}{2^n}<\varepsilon$, and let $y\in B_{x,n+1}$. Then by (*), $p_u(x, y)=\sum_{i=1}\frac{1}{2^i}p_{B_{u,i}}(x, y)\leqq\frac{1}{2^{n+2}}+\frac{1}{2^{n+3}}+\cdots<\frac{1}{2^n}<\varepsilon$ for each $u\in X$, and hence $p(x, y)<\varepsilon$. Thus $\mathfrak{I}$ is finer than $\mathfrak{I}_p$.

Remark. In the paper [3], M. Sion and G. Zelmer proved that a topological space having a σ-point finite base is quasi-pseudo-metrizable. In the proof of theorem 1, we constructed a σ-point finite base, hence theorem 1 is proved by use of M. Sion and G. Zelmer's result. But our purpose is to prove the theorem by direct construction.

Theorem 2. *Let $(X,\mathfrak{J})$ be a first countable space, and*

$X_1=\{x\in X|\mathfrak{J}_x$ *has no minimal member with respect to* $\subset\}$,

where $\mathfrak{J}_x=\{O\in\mathfrak{J}|x\in O\}$ *for each* $x\in X$.

If X_1 is a finite set and $\cup\{V_x|x\in X-X_1\}\subset X-X_1$ *(where V_x is the minimal member of $\mathfrak{J}_x$), then $(X,\mathfrak{J})$ is quasi-pseudo-metrizable.*

Proof. For each $x\in X_1$, let $\{V(x,i)|i=1,2,\cdots\}$ be a open local base of x.

Define $p_u(x,y)=\frac{1}{2}p_{V_u}(x,y)$, if $u\notin X_1$, where V_u is the minimal member of $\mathfrak{J}_u$, and for $A\subset X$, p_A is a quasi-pseudo-metric on X defined in the proof of theorem 1, and define

$$p_u(x,y)=\sum_{i=1}\frac{1}{2^i}p_{V(u,i)}\ (x,y)\ \text{if}\ u\in X_1$$

and $$p(x,y)=\sup\{p_u(x,y)|u\in X\}.$$

Let $x\in U\in\mathfrak{J}$. If $x\notin X_1$, then $S_p(x,\frac{1}{2})\subset U$. For, $p(x,y)<\frac{1}{2}$ implies $y\in V_x$ since $p_x(x,y)=\frac{1}{2}p_{V_x}(x,y)<\frac{1}{2}$. If $x\in X_1$, choose $V(x,n)$ such that $x\in V(x,n)\subset U$. It follows that $S_p(x,\frac{1}{2^n})\subset U$ because $p(x,y)<\frac{1}{2^n}$ implies

$$p_x(x,y)=\sum_{i=1}\frac{1}{2^i}p_{V(x,i)}\ (x,y)<\frac{1}{2^n}.$$

Thus $\mathfrak{J}$ is coaser than $\mathfrak{J}_p$.

Let $x\in S_p(x,r)$, $r>o$.

If $x\notin X_1$, $V_x\subset S_p(x,r)$, where V_x is the minimal member of $\mathfrak{J}_x$.

If $x\in X_1$, let $\frac{1}{2^n}<r$, and choose $V(x,1)$ such that

$$V(x,1)\subset\cap\{V(u,j)|x\in V(u,j),\ u\in X_1,\ j=1,2,\cdots,\ n+1\}.$$

Then $V(x,1)\subset S_p(x,r)$. For $y\in V(x,1)$ implies

$$p_u(x,y)=\sum_{i=1}\frac{1}{2^i}p_{V(u,i)}(x,y)\leq\frac{1}{2^{n+2}}+\frac{1}{2^{n+3}}+\cdots<\frac{1}{2^n}<r$$

for all $u\in X_1$,

$$p_u(x,y)=\frac{1}{2}p_{V_u}(x,y)=0\ \text{for all}\ u\notin X_1,\ \text{since}\ x\notin V_u,$$

and hence $p(x,y)<r$. Thus $\mathfrak{J}_p$ is coaser than $\mathfrak{J}$.

The following example shows that there exists a topological space $(X,\mathfrak{J})$ having the condition in theorem 2 which has no σ-point finite base.

Example. Let $X_1=\{a|a=\text{ordinal},\ a<\Omega\}-\{\omega\cdot1,\ \omega\cdot2,\ \cdots,\ \omega\cdot n\}$, where Ω : the first uncountable ordinal, n : natural number. Choose $a_1,a_2,\cdots,a_n$ ($a_i\neq a_j$ if $i\neq j$) such that

$$X_1\cap\{a_1,a_2,\cdots,a_n\}=\phi.$$

Let $X=X_1\cup\{a_1,a_2,\cdots,a_n\}$, and

$$\mathcal{A}_1=\{[i,\omega)|i<\omega\},$$

$$\mathcal{A}_j=\{[\omega\cdot(j-1)+i,\ \omega\cdot j)|1\leq i<\omega\};\ j=2,3,\cdots,n,$$

$$\mathcal{A}'_j=\{A\cup\{a_j\}|A\in\mathcal{A}_j\};\ j=1,2,\cdots,n,$$

$$\mathfrak{L}_1=\{[a,\Omega)|(\omega\cdot n)+1\leq a<\Omega\},$$

and let $\mathfrak{L}=\mathfrak{L}_1 \cup [\bigcup_{j=1}^{n} \mathcal{a}_j] \cup [\bigcup_{j=1}^{n} \mathcal{a}'_j]$

Let $\mathfrak{J}$ be the topology on X having $\mathfrak{L}$ as a base. Then $\mathfrak{J}$ satisfies the condition in theorem 2 but $\mathfrak{J}$ has *no* σ-point finite base.

REFERENCES

[1] J.L. Kelley, (1955), *General Topology*, Van Nostrand, Princeton,

[2] W.J. Pervin (1964), *Foundations of General Topology*, Academic Press, New York.

[3] M. Sion and G. Zelmer (1967), On quasi-metrizability, *Canadian J. of Math.*, Vol. XIX, **6**. 1234-1249

[4] C.W. Patty (1967), Bitopological Spaces, *Duke Math. J.*, Vol. 34, **3**.

[5] S. Willard (1970), *General Topology*, Addison-Wesley, Reading.