Distribution of Success Runs for Markov Dependent Bernoulli Trials

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ABSTRACT

For a sequence of Markov dependent Bernoulli trials, asymptotic normality of the number of success runs with length $r \geq 1$ is established, when the number of trials tends to infinity.

1. Introduction and Model

Consider a sequence of Bernoulli trials $X_1, X_2, \ldots,$ where

$$P[X_i=1]=1-P[X_i=0]=p=1-q$$

with Markov dependence such that

$$P[X_i=1 \mid X_{i-1}=1]=\lambda.$$

As a consequence it follows that

$$P[X_i=0 \mid X_{i-1}=1]=1-\lambda$$

$$P[X_i=1 \mid X_{i-1}=0]=\eta$$

$$P[X_i=0 \mid X_{i-1}=0]=1-\eta, \text{ where } \eta=(1-\lambda)p/q$$

The condition $\max(0,(2p-1)/p) \leqq \lambda \leqq 1$ is imposed so that the probabilities are non-negative. This model is a generalization of the usual independent Bernoulli trials [2] and [3] and reduce to it for $\lambda=p$.

In this paper the event $X_i=1$ referred to as the ith trial is a success and we study the distribution of success runs of length $r \geq 1$. we adopt the defini-

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tion of success runs of length r given in Feller [1, p. 264]. That is, "A sequence of n Bernoulli trials contains as many runs of length r as there are nonoverlapping uninterrupted succession of exactly r successes. In a sequence ro Bernoulli trials, a run of length r occurs at nth trial, if the nth trial adds a new run to the sequence."

Let E be the event that there is a run of successes with length r in a sequence of Markov dependent Bernoulli trials. Let u_n denote the pobability E occurs at nth trial and f_n denotes the probability that E occurs at nth trial for the first time. For the case of independent Bernoulli trials, i.e., $\lambda=p$, probability generating functions of u_n and f_n are obtained in Feller [1, p. 264—266]. We obtain the generating functions for $r=1$ in section 2 and in section 3 the generating functions for $r\geq 2$. Let $N_{n,r}$ denote the number of success runs with length r produced in n trials. The limiting distribution of $N_{n,r}$ as $n\rightarrow\infty$, is considered in section 4.

2. Probability Generating Functions of Success Runs with Length One

Let $v_n=\text{p}[X_{n+1}=1\,|\,X_1=1]$,

$q_n=\text{p}[X_2=X_3=\ldots=X_n=0,\ X_{n+1}=1\,|\,X_1=1]$,

$u_n=\text{p}[X_n=1]$, and

$f_n=\text{p}[X_1=X_2=\ldots=X_{n-1}=0, X_n=1]$.

Consequently, we have for $n\geq 1$

$$u_n=v_0 f_n+v_1 f_{n-1}+\ldots+v_{n-1}f_1$$
$$v_n=v_o g_n+v_1 g_{n-1}+\ldots+v_{n-1}g_1, \qquad (2.1)$$

with $u_o=v_o=1$ and $f_o=g_o=0$.

Now multiply (2.1) by s^n and sum over $n=1, 2, \cdots$. We get, respectively,

$$[U(s)-1]=F_1(s)\ V(s)$$
$$[V(s)-1]=G(s)\ V(s),\ \text{where} \qquad (2.2)$$

$$U(s)=\sum_{j=0}^{\infty}u_j s^j,\ \ V(s)=\sum_{j=0}^{\infty}v_j s^j$$

$$F_1(s)=\sum_{j=1}^{\infty}f_js^j,\quad G(s)=\sum_{j=1}^{\infty}g_js^j.$$

Eliminating $V(s)$ in (2.2) we get

$$U(s)=1+\frac{F_1(s)}{1-G(s)}. \tag{2.3}$$

Since $U(s)=1+\frac{ps}{1-s}$, we have

$$\left(\frac{1-s}{ps}\right)F_1(s)+G(s)=1. \tag{2.4}$$

Using the definition of f_n and (2.4), we have

$$F_1(s)=ps+\frac{q\eta s^2}{1-(1-\eta)s}, \tag{2.5}$$

$$G(s)=\lambda s+\frac{(1-\lambda)\eta s^2}{1-(1-\eta)s}. \tag{2.6}$$

It can be shown that f_n and g_n obtained from (2.5) and (2.6) agree with the results given by Klotz and Park [4]. That is

$f_1=p$

$f_n=q\eta(1-\eta)^{n-2}$ for $n=2, 3, \ldots,$ and

$g_1=\lambda$

$g_n=(1-\lambda)\eta(1-\eta)^{n-2}$ for $n=2, 3, \ldots$.

3. Probability Generating Functions of Success Runs with Length $r=2$

In this section we adopt the definition of u_n and f_n given in section 1. We have for $n\geq r$,

$$P[X_{n-r+1}=1,\ X_{n-r+2}=1, \ldots,\ X_{n-1}=1,\ X_n=1]=p\lambda^{r-1}.$$

In this case E is a recurrent event. The probability that E occurs at trial number $n-k$ $(k=0, 1, \ldots, r-1)$ and the following k trials result in success is $u_{n-k}\lambda^k$. Since these r possibilities are mutually exclusive we get, for $n\geq r$,

$$u_n+u_{n-1}\lambda+\ldots+u_{n-r+1}\lambda^{r-1}=p\lambda^{r-1}, \quad \text{where} \tag{3.1}$$

$$u_1=u_2=\ldots=u_{r-1}=0 \text{ and } u_o=1.$$

Multiply (3.1) by s^n and sum over $n=r, r+1, \ldots$. We get on the left side

$[U_r(s)-1]\ [1-(\lambda s)^r]/(1-\lambda s)$ and on the right hand side $p\lambda^{r-1}s^r/(1-s)$, where

$U_r(s)=\sum_{j=0}^{\infty}u_j s^j$. Consequently

$$U_r(s)=1+\frac{p\lambda^{r-1}s^r(1-\lambda s)}{(1-s)[1-(\lambda s)^r]}.$$

Let $F_r(s)=\sum_{j=1}^{\infty}f_j s^j$. Since

$F_r(s)=1-\dfrac{1}{U_r(s)}$, we have

$$F_r(s)=\frac{p\lambda^{r-1}s^r(1-\lambda s)}{p\lambda^{r-1}s^r(1-\lambda s)+(1-s)[1-(\lambda s)^r]}. \tag{3.2}$$

Note that $F_r(s)$, when $\lambda=p$, coincides with the generating function given in Feller [1, p. 264].

4. The Limiting Distribution of Recurrence Times of Success Runs with Length r

Let $N_{n,r}$ denote the number of success runs with length r produced in n consecutive Markov dependent Bernoulli trials. Let $S_{k,r}$ denote the number of trials up to and including the kth occurrence of E. Then we have

$$\mathrm{p}[N_{n,r}\geq k]=\mathrm{p}[S_{k,r}\leq n]. \tag{4.1}$$

For $r\geq 2$, E is a certain recurrent event, thus we have

$S_{k,r}=Y_{1,r}+Y_{2,r}+\ldots+Y_{k,r}$, where $Y_{j,r}$'s are mutually independent identically distributed random variables with common probability generating function given by (3.2). Hence the central limit theorem can be used to establish

$$\frac{S_{k,r}-kE[Y_{j,r}]}{\sqrt{kV(X)}}\xrightarrow{D}N(0,1)\ \text{as}\ k\to\infty, \tag{4.2}$$

where $N(0,1)$ is a standard normal random variable and $\xrightarrow{D}$ denotes convergence in distribution.

Similarly, for $r=1$, we can write

$S_{k,1}=Y_{1,1}+Y_{2,1}+\ldots+Y_{k,1}$, where

$Y_{1,1}$ has probability generating function given by (2.5) and $Y_{j,1}$, $j=2,\ \ldots,k$ are identically distributed with the common probability generating function given by (2. 6) and $Y_{j,1}$'s are mutually independent. Hence the central limit theorem can be used to establish

$$\frac{S_{k,1}-[E(Y_1)+(k-1)E(Y_2)]}{\sqrt{V(Y_1)+(k-1)V(Y_2)}} \xrightarrow{D} N(0,1) \text{ as } k\to\infty \tag{4.3}$$

Now using (2. 4), (2. 5) and (2. 6) we have

$$F_1'(1)=(1-p+\eta)/\eta, \quad F''(1)=2(1-p)/\eta^2$$

$$G'(1)=1/p, \qquad G''(1)=2(1-\lambda)/\eta^2$$

Using (3. 2) we have

$$F_r'(1)=\lambda(1-\lambda^r)/[p\lambda^r(1-\lambda)] \text{ and}$$

$$F_r''(1)=-2r\lambda^r/[p\lambda^{r-1}(1-\lambda)]+2(\lambda^r-1)\,[rp\lambda^{r-1}-(r+1)\,p\lambda^r]/[p\lambda^{r-1}(1-\lambda)]^2+2\{(1-\lambda^r)/[p\lambda^{r-1}(1-\lambda)]\}^2.$$

Consequently, after some algebra

$$E(Y_{1,1})=(1-p+\eta)/\eta,$$

$$\mu_1=E(Y_{j,1})=1/p \quad \text{for } j\geq 2,$$

$$V(Y_{1,1})=F_1''(1)+F_1'-[F_1'(1)]^2=\frac{(1-p)(1+p-\eta)}{\eta^2}.$$

$$\sigma_1^2=V(Y_{j,1})=G''(1)+G'(1)-[G'(1)]^2=\frac{(1-\lambda)(2p-\eta)}{p\eta^2}.$$

For $r\geq 2$,

$$\mu=E(Y_{j,r})=F_r'(1)=\lambda(1-\lambda^r)/[p\lambda^r(1-\lambda)]=\frac{1-\lambda^r}{\lambda^r(1-\lambda)}-\frac{(p-\lambda)(1-\lambda^r)}{p\lambda^r(1-\lambda)}.$$

$$\sigma_r^2=V(Y_{j,r})=F_r''(1)+F_r'(1)-[(F_r'(1)]^2. \tag{4.4}$$

Substituting $F'(1)$, $F''(1)$ in (4. 4) and simplifying, we have

$$\sigma_r^2=\frac{\lambda^2}{[p\lambda^r(1-\lambda)]^2}-\frac{2r\lambda}{p\lambda^r(1-\lambda)}-\frac{\lambda^2(1-p)}{p^2\lambda^r(1-\lambda)^2}-\frac{\lambda^2}{p(1-\lambda)^2}+\frac{(p-\lambda)}{p^2\lambda^{r-1}(1-\lambda)^2}(1-\lambda^r).$$

After some algebra, it can be written

$$\sigma_r^2=\frac{1}{[\lambda^r(1-\lambda)]^2}-\frac{2r+1}{\lambda^r(1-\lambda)}-\frac{\lambda}{(1-\lambda)^2}+\frac{(p-\lambda)}{p^2\lambda^{2r}(1-\lambda)^2}[2rp\lambda^r(1-\lambda)+(1-p)\lambda^{r+1}(1-\lambda^r)-(p+\lambda)(1-\lambda^r)].$$

Note that σ_r^2 reduces to the expression given in Feller [1, pp. 264] when $\lambda=p$.

Now using (4. 1) and (4. 2) we have for $r \geq 2$

$$\frac{[N_{n,r}-n/\mu]}{\sqrt{n}\ \sigma_r/\mu^{3/2}} \xrightarrow{D} N(0, 1) \text{ as } n\rightarrow\infty.$$

Similarly using(4. 1) and(4. 3) we have

$$\frac{[N_{n,1}-n/\mu_1]}{\sqrt{n}\ \sigma_1/\mu_1^{3/2}} \xrightarrow{D} N(0. 1) \text{ as } n\rightarrow\infty.$$

This result coincides with that of Klotz [3] and Gabriel [2] who used somewhat different models than ours.

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