Dynamics of Heterogeneous Warfare

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There have been many modeling attempts in the past to formulate combat operations analytically. Any attempt to formulate the problem mathematically necessarily leaves out many relevant factors. The attempt is nevertheless illuminating, for two reasons. First, it provides an insight into the effect of various weapon systems characteristics and tactical strategies on the outcome of a hypothetical warfare. Second, it suggests what parameters need to be measured to understand the dynamics of a warfare.

One of the earliest examples of military operational research is Lanchester's study of combat. Later, Koopman extended Lanchester's results and suggested a reformulation of the problem in stochastic form [7]. Since then many researchers refined and extended the fundamental model to study the microscopic aspect of a duel such as firing rates of the duelists, varying single-shot kill probability of the rounds [2], and other situational characteristics such as ammunition limitation [1], surprise [9], cover and concealment [3], etc..

In the probabilistic development of the fundamental Lanchester theory, some of the desired results are (1) the probability that having started with x_0 and y_0 units, x and y units will be still in operation a time t later and (2) probability that x wins [4, 9].

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The difficulties encountered in the stochastic analysis of combat are generally of a computational nature. It may be possible to solve the partial difference-differential equations that arise, but the solutions are not especially useful. What seems to be required is a simple measure of effectiveness of the correct result.

Theory of fundamental duels provides an invaluable macroscopic insight into the role played by the basic parameters and combat strategies. However, frequently more than one kind of weapon system is employed by each side let alone the differential performances among the weapons of one type.

One of the approaches to tackle the problem of heterogeneity is a discrete-event computer simulation. Fain, et al [5], divided the forces into offensive units and defensive units. The simulation program treats attrition of the engaged units by Lanchester-like equations. The attrition rate suffered by a unit is proportional to the fire power directed against it by opposing offensive units with which it is engaged.

Generally, the formulation of a simulation model often requires consideration of more details than necessary in an analytic model, some of which might not be crucial for the analysis of the problem. Analytic model is more explicit about the relations between parameters, and can provide important knowledge on how results depend on certain parameter combinations.

Several authors including Weiss [8] considered various cases of heterogeneous forces. In most cases, it is assumed that the attrition expressions are *deterministic* and of a linear form. Under this assumption, a generalized Lanchester-type square law holds for the heterogeneous case. The linear attrition expression applies to the situation where fire is shifted to a new target when a target is destroyed, or the area held is released progressively as the force dwindles in size so that the density of the forces in the area remains constant [3].

Statement of the Problem

Two forces, X and Y, both with heterogeneous multiple weapon systems engage in combat. Each weapon with given characteristics fires at a given rate, which is assumed to correspond to a Poisson stream. It is assumed that there are no replacements, so that the only changes in the strengths of the forces are decrements.

Each force has information about the general areas in which opposing units are located. Indirect, area fire is used by both sides which defend their area occupied initially so that density of the forces in the area decreases as the battle progresses. Thus neither side concentrates fire on the opponent survivors.

Depending on the availability of spontaneous information on the current enemy status, each participating unit may follow (1) a prescribed attack pattern in terms of the fraction of the available units (or the fraction of fire from the available units) allocated to various enemy targets, or (2) an adaptive attack pattern depending on the enemy status at that time.

As the struggle goes on, the number of survivors on the two forces will tend to diminish. If the total number of a force reaches zero, the other force is said to win the battle. However, for the heterogeneous weapons systems, there can be a stalemate depending on the two forces' weapon allocation strategies, which is generally determined from the information on the current enemy status.

The problem addressed in this paper is to determine the expected number of survivors of various weapon types at any given time.

Formulation and Solution of the Problem

The problem proposed in the preceding section can be described more accurately with the following symbols:

$x_i = x_i(t)$

$=$ number of weapons, type i of side X, at time t; $(1 \leq i \leq M)$

$y_j = y_j(t)$

$=$ number of weapons, type j of side Y, at time t; $(1 \leq j \leq N)$

$\%^x_{ij} = \%^x_{ij}(y_1(t), y_2(t), \ldots, y_N(t), t)$

$=$ fraction of x_i allocated to targets y_j $\left(\sum_{j=1}^{N} \%^x_{ij} = 1\right)$

$\%^y_{ji} = \%^y_{ji}(x_1(t), x_2(t), \ldots, x_M(t), t)$

$=$ fraction of y_j allocated to targets x_i $\left(\sum_{i=1}^{M} \%^y_{ji} = 1\right)$

$r^x_i =$ firing rate of each x_i

$r^y_j =$ firing rate of each y_j

$\pi^x_{ij} =$ single-shot kill probability (expected rate at which a weapon system can destroy targets) of each x_i firing on single "y_j"

$\pi^y_{ji} =$ single-shot kill probability of each y_j firing on single "x_i"

To formulate the problem the following state probabilities are defined:

$p^x_i(m,t) = \text{Prob}\{x_i(t) = m\}$

$p^y_j(n,t) = \text{Prob}\{y_j(t) = n\}$

Then, by definition the expected number of survivors of various weapon types for both sides are:

$$\overline{x_i}(t) = \sum_{m=0}^{x_i(0)} m \cdot p^x_i(m,t); \qquad 1 \leq i \leq M \tag{1}$$

and

$$\overline{y_j}(t) = \sum_{n=0}^{y_j(0)} n \cdot p^y_j(n,t); \qquad 1 \leq j \leq N \tag{2}$$

Furthermore, the following transition probabilities are defined based on the *assumption that all weapon systems are deployed over the geographical area in such a way that the chance of multiple destruction from a single enemy shot is negligible*:

$\Delta p^x_i x_i(m,t)\,dt = \text{Prob}\{x_i(t+dt) = m \mid x_i(t) = m-1\}.$

In addition to this basic assumption, also note that the *probability of more than two enemy weapons firing simultaneously in a small time interval dt approa-*

ches zero as dt approaches 0 because each weapon's firing process is independent of the other's.

Connecting the various state probabilities at time t and $t+dt$, the following difference-differential equation is obtained describing the attrition of X force.

$$\begin{aligned} p_i^x(m,t+dt) &= p_i^x(m+1,t)\cdot \Delta p_i^x(m+1,t)dt + p_i^x(m,t)\cdot[1-\Delta p_i^x(m,t)dt] \\ &\quad +0(dt); \quad 0\leq m\leq x_i(0)-1 \\ &= p_i^x(m,t)\cdot[1-\Delta p_i^x(m,t)dt]+0(dt); \qquad m=x_i(0) \end{aligned} \tag{3}$$

But

$$\Delta p_i^x(m,t)dt = \sum_{j=1}^{N}\text{Prob}\left\{\begin{matrix}\text{one out of } m \text{ “}x_i\text{” killed} \\ \text{by “}y_j\text{” in } dt\end{matrix}\right\}$$

$$= \sum_{j=1}^{N}\sum_{n=0}^{y_j(0)}\text{Prob}\left\{\begin{matrix}\text{one out of } m \text{ “}x_i\text{” killed} \\ \text{by any of “}y_j\text{” in } dt\end{matrix}\,\middle|\, y_j=n\right\}\cdot p_j^y(n,t)$$

$$= m\left(\sum_{j=1}^{N}\sum_{n=0}^{y_j(0)} n\ \%_{ji}^y\ r_j^y dt \pi_{ji}^y\right)p_j^y(n,t) \tag{4}$$

Equation (4) arises naturally under the Poisson assumption of the firing process. In a small time interval dt the chance of destruction is proportional to the attacking enemy's $(n\cdot\%_{ji}^y\ r_j^y dt)$ effective kill rate (π_{ji}^y)[1], and to the current number of units (m) since and one of m units might be hit. Equation (4) can be simplified further by substituting Equation (2).

$$\Delta p_i^x(m,t)dt = m\sum_{j=1}^{N}\%_{ji}^y\ r_j^y\ \pi_{ji}^y \overline{y}_j(t)dt \tag{5}$$

Transposing and taking $\lim_{dt\to 0}$ of both sides of Equation (3) after substituting Equation (5),

1) The reason for this proportionality is intuitive and simple. Since Prob{destruction in dt} = Prob{fire in dt} × Prob{hit}, when $n=100$, $\%=0.5$, $r=60/\text{min}$ (say), and $dt=\frac{1}{300,000}$ min (say), firing occurs once in every 100 time intervals on the average, i.e., the probability that a firing occurs in a specific time interval dt is only $\frac{1}{100}$. If n doubles ($n=200$), firing occurs once in every 50 time intervals on the average and the probability that a firing occurs in a specific time interval dt doubles($\frac{1}{50}$), too. Furthermore, the size of the infinitesimal time interval dt is only a conceptual one and its limit is 0.

$$\frac{d}{dt}p_i^x(m,t)=\{(m+1)p_i^x(m+1,t)-mp_i^x(m,t)\}\cdot$$

$$\{\sum_{j=1}^{N}\%_{ji}^{y}\ r_j^y\ \pi_{ji}^y\overline{y_j}(t)\};\qquad 0\leq m\leq y_i(0)-1$$

$$=-mp_i^x(m,t)\sum_{j=1}^{N}\%_{ji}^{y}\ r_j^y\ \pi_{ji}^y\overline{y_j}(t);\ \ m=x_i(0) \tag{6}$$

But since

$$\overline{x_i}(t)=\sum_{m=0}^{x_i(0)}m\cdot p_i^x(m,t)$$

$$\frac{d}{dt}\overline{x_i}(t)=\sum_{m=0}^{x_i(0)}m\cdot\frac{d}{dt}p_i^x(m,t)$$

$$=\{-x_i(0)^2\cdot p_i^x(x_i(0),t)+\sum_{m=0}^{x_i(0)-1}m[(m+1)p_i^x(m+1,t)-mp_i^x(m,t)]\}\cdot$$

$$\sum_{j=1}^{N}\%_{ji}^{y}\ r_j^y\ \pi_{ji}^y\overline{y_j}(t)$$

$$=\{-p_i^x(1,t)+2p_i^x(2,t)-4p_i^x(2,t)+\cdots+[x_i(0)-1][x_i(0)]p_i^x(x_i(0),t)$$

$$-x_i(0)^2p_i^x(x_i(0),t)\}\sum_{j=1}^{N}\%_{ji}^{y}\ r_j^y\ \pi_{ji}^y\overline{y_j}(t)$$

$$=-\left\{\sum_{m=0}^{x_i(0)}mp_i^x(m,t)\right\}\ \sum_{j=1}^{N}\%_{ji}^{y}\ r_j^y\ \pi_{ji}^y\overline{y_j}(t)$$

$$=-\overline{x_i}(t)\sum_{j=1}^{N}\%_{ji}^{y}\ r_j^y\ \pi_j^y\ \overline{y_j}(t) \tag{7}$$

Similarly, the attrition of Y force is described by;

$$\frac{d}{dt}\overline{y_j}(t)=-\overline{y_j}(t)\sum_{i=1}^{M}\%_{ij}^{x}r_i^x\pi_{ij}^{x}\ \overline{x_i}(t) \tag{8}$$

These differential equation models of warfare are deterministic in a sense, always yielding the same result for given initial conditions. However, such attrition equations are *derived via stochastic formulation* to represent the mean course of battle having an underlying probability distribution.

Fortunately, there are many numerical techniques available to solve the system of differential equations of this type. Some of the solution techniques are preprogrammed as computer subroutines and readily available (e.g., subroutines "RKGS" and "HPCG" in FORTRAN).

Lack of Information on the Enemy Status and Equilibria

The first point we may observe about the system of Equations (7) and (8) is that they define certain conditions of equilibrium or stalemate. That is to say whenever

$$\frac{d}{dt}\overline{x_i}(t)=\frac{d}{dt}\overline{y_j}(t)=0 \qquad \text{for all } i \text{ and } j.$$

We thus have $M+N$ equations between the M+N variables $\overline{x_i}(t)$ and $\overline{y_j}(t)$, which determine certain values

$$\overline{x_i}(t)=c_i \quad \text{and} \quad \overline{y_j}(t)=c_j \qquad \text{for all } i \text{ and } j.$$

In general there may be a number of such possible equilibria.

The outcome of the battle (terminal state) depends on the specific initial condition and weapon allocation strategy of the forces. However, it is entirely possible that under certain weapon allocation strategies the initial condition of the forces may correspond to an equilibrium state; or during the course of the battle the number of survivors on two sides dwindle to an equilibrium state resulting in a stalemate.

Physically, a stalemate can arise if each participating unit follows a prescribed, non-adaptive attack pattern when each force does not have spontaneous information on the current enemy status.

One way to avoid a stalemate is to make the weapon allocation strategy adaptive to the enemy status. That is

$$\%_{ij}^{x}=\%_{ij}^{x}(y_1(t),y_2(t),\ldots,y_N(t),t)$$

and

$$\%_{ji}^{y}=\%_{ji}^{y}(x_1(t),x_2(t),\ldots,x_M(t),t)$$

State Equations

The system of equations may be readily solved either with or without the independent variable, t, eliminated from the equations. The solutions achieved with t first eliminated as a variable may be called the state equations, wherein the state of either force at any given stage of the combat is simply specified in terms of the other's state.

For the brevity of notation, let $c^x_{ij} = \%^x_{ij}\, r^x_i \pi^x_{ij}$ and $c^y_{ji} = \%^y_{ji}\, r^y_j \pi^y_{ji}$. From Equations (7) and (8),

$$\sum_{i=1}^{M} c^x_{ij} \frac{d}{dt} \overline{x_i}(t) = -\sum_{i=1}^{M} \sum_{j=1}^{N} c^x_{ij}\, c^y_{ji}\, \overline{x_i}(t)\, \overline{y_j}(t) \tag{9}$$

$$\sum_{j=1}^{N} c^y_{ji} \frac{d}{dt} \overline{y_j}(t) = -\sum_{j=1}^{N} \sum_{i=1}^{M} c^y_{ji}\, c^x_{ij}\, \overline{y_j}(t)\, \overline{x_i}(t) \tag{10}$$

The equality of the right hand sides of Equatinos (8) and (9) leads to the following system of $M \times N$ equations:

$$\sum_{i=1}^{M} c^x_{ij}[x_i(0) - \overline{x_i}] = \sum_{j=1}^{N} c^y_{ji}[y_j(0) - \overline{y_j}] \qquad \text{for all } i \text{ and } j. \tag{11}$$

The Equation (11) specifies the dynamic states in terms of the surviving X-force $(\overline{x_i})$ corresponding to a specified terminal state of the Y-force $(\overline{y_j})$, and vice versa.

Numerical Demonstration

Consider a battle between two forces with the characteristics described in Table 1. Let $\dot{x}_i$ denote $\frac{d}{dt}\overline{x_i}(t)$ for brevity of notation. From Equations (7) and (8), the following systems of differential equations are obtained.

(1) X-strategy I (non-adaptive)

$\dot{x}_1 = -x_1(.0015y_1 + .001y_2)$

$\dot{x}_2 = -x_2(.002y_1 + .002y_2)$

$\dot{y}_1 = -y_1(.003x_1)$

$\dot{y}_2 = -y_2(.004x_2)$

Table 1. Weapon Characteristics and Strategies for the Numerical Examples

Weapon Characteristics	Both Sides	
Weapon Type	1 (heavy)	2 (light)
Firing Rate/min.	.1	.2
Initial Size	50	100

Weapon Type	Target Type	Single-shot Kill Probability (both sides)	Weapon Allocation: X-strategy I	Weapon Allocation: X-strategy II	Weapon Allocation: Y-strategy
1	1	.03	100%	$\frac{y_1}{y_1+y_2}$	50%
	2	.04	0%	$\frac{y_2}{y_1+y_2}$	50%
2	1	.01	0%	$\frac{y_1}{y_1+y_2}$	50%
	2	.02	100%	$\frac{y_2}{y_1+y_2}$	50%

(2) X-strategy II (adaptive)

$$\dot{x}_1 = -x_1(.0015y_1 + .001y_2)$$

$$\dot{x}_2 = -x_2(.002y_1 + .002y_2)$$

$$\dot{y}_1 = -y_1\left(\frac{.003y_1}{y_1+y_2}x_1 + \frac{.002y_1}{y_1+y_2}x_2\right)$$

$$\dot{y}_2 = -y_2\left(\frac{.004y_2}{y_1+y_2}x_1 + \frac{.004y_2}{y_1+y_2}x_2\right)$$

The systems of differential equations are solved by the commonly available subroutine "RKGS" in FORTRAN and the results are displayed in Figure 1.

Connection with Lanchester's Model

Probably one of Lanchester's contributions to the study of combat is his distinguishing between warfare that consists of a series of duels without concentration of power (Linear Law), and warfare in which each side is able to concentrate its entire force on the other (Square Law).

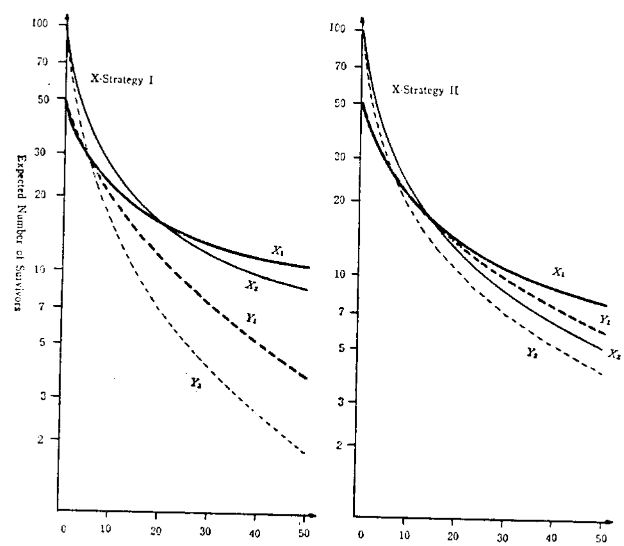
If both sides possess only one type of weapon, the model developed in this paper indeed reduces to

$$\dot{x} = -(r\pi)_y xy$$
$$\dot{y} = -(r\pi)_x xy$$

which are commonly known as Lanchester's Linear Law. Differential equations of this sort have been discussed to describe the struggle between animal species [6]. Lanchester also noted these nonlinear equations, but he did not develop the consequences of the theory behind them [3].

Conclusion

The model provides a convenient analytical tool wherever questions concerning the trade-offs between such parameters as weapon accuracy, lethality, rates of fire, etc. are important. A second area of application is in evaluating tactics and strategies. The model can answer the questions such as: (1) Should a weapon system be exclusively assigned to attack a certain enemy target or used as a general support? (2) Would a certain condition lead to a stalemate?



Duration of Engagement (minutes)

Figure 1

Effect of Weapon Allocation Strategies

ABSTRACT

The relative importance of single-shot kill probabilities, rates of fire, weapon allocation strategies, and the size of initial force in warfare between two forces with heterogeneous multiple weapon systems are considered by examining their effect on a natural measure of effectiveness, the expected number of survivors. *Attrition equations are derived via stochastic formulation* to represent the mean course of battle having an underlying probability distribution.

It is assumed that each side uses indirect area fires. Level of intelligence activities are reflected in the availability of spontaneous information on the current enemy status. Depending on the availability of the information on the current enemy status, each participatory unit may follow

1) a prescribed attack pattern (fraction of the available units assigned to various enemy targets) or
2) an adaptive attack pattern depending on the enemy status at that time.

Conditions for possible stalemate are discussed.

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