Finite Element Method in Structural Analysis of Ships

by

Yoshiyuki Yamamoto*

Introduction

The finite element method has been developed since late 1950's, and it has been applied to various fields of structural analysis. As for ships, it was first applied to structural analysis by Det norske Veritas[7], who developed a computer program system SESAM 69 later for this purpose.

The finite element method is an approximation method which can be easily understood by biginners. It is based on the well-established variational principle of elasticity; therefore, any boundary condition can be approximated from the view-point of energy principles, and bounds of errors can be estimated with the aid of functional analysis.

Computer programs for the finite element method can easily be developed: A biginer can write a program for analysis of a simple two-dimensional problem, and its scheme is the same as that of the main part of a large-scaled general-purpose program.

The finite element method started with linear elastic stress analysis. With the progress of computers, the range of its application has been expanded to elasto-plastic stress analysis, buckling and geometrically non-linear problems, dynamic behaviors of structures, interaction problems of structures with their environments, simulation of a structure under varying conditions, and design optimization of structures. Recently it has been applied to problems other than structural analysis on the basis of an appropriate variational or some other principle.

Finite Element Method [1-3]

In the case of linear elasticity, there are many variational principles, and from each of them, the respective finite element formulation can be established. In the following, the stiffness method, the formulation based on the principle of minimum potential energy, will be mainly discussed.

The procedure of the stiffness method applied to linear elasticity consists of the following six steps;

1. decompose a structure under consideration into small elements with simple typical shapes, and define the nodal displacements;
2. introduce shape functions for each element so that they may be compatible along the inter-element boundaries, and define coordinate functions corresponding to nodal displacements;
3. derive the element stiffness matrices k corresponding to nodal displacements, and obtain the load vectors f;
4. assemble elements stiffness matrices and load vectors to obtain the global stiffness matrix K and the global load vector F, which give the equilibrium equation;
5. solve the equilibrium equation given as a set of linear algebraic equations; and
6. calculate the stress distributions in each element.

This procedure can be understood by Fig.1. In the case of nonlinear problems, step 3 is modified at each loading step, and the rest of procedure is repeatedly applied according to the loading history. For elasto-plastic analysis, elastic modulus tensors are modified by taking into consideration of occurrence of plastic loading and unloading processes. In the case of dynamic problems, a mass matrix should be introduced. For numerical simulations, it is convenient that the structure is decomposed in conformity to the process of simulation.

With the aid of functional analysis, it can be pro-

* Professor, Department of Naval Architecture, University of Tokyo. Lecture delivered at the Special Lecture Conference for Ship Structure on April 21, 23, 1977

ved that finite element solutions converge to the rigorous one with the decrease of the element size. Errors of finite element solutions of this kind are called *discretization errors*, and are in proportion to square of the element size. Numerals treated on an electric digital computer are represented in the binary system of a finite number of bits, and it causes numerical errors at each process of computation. Errors

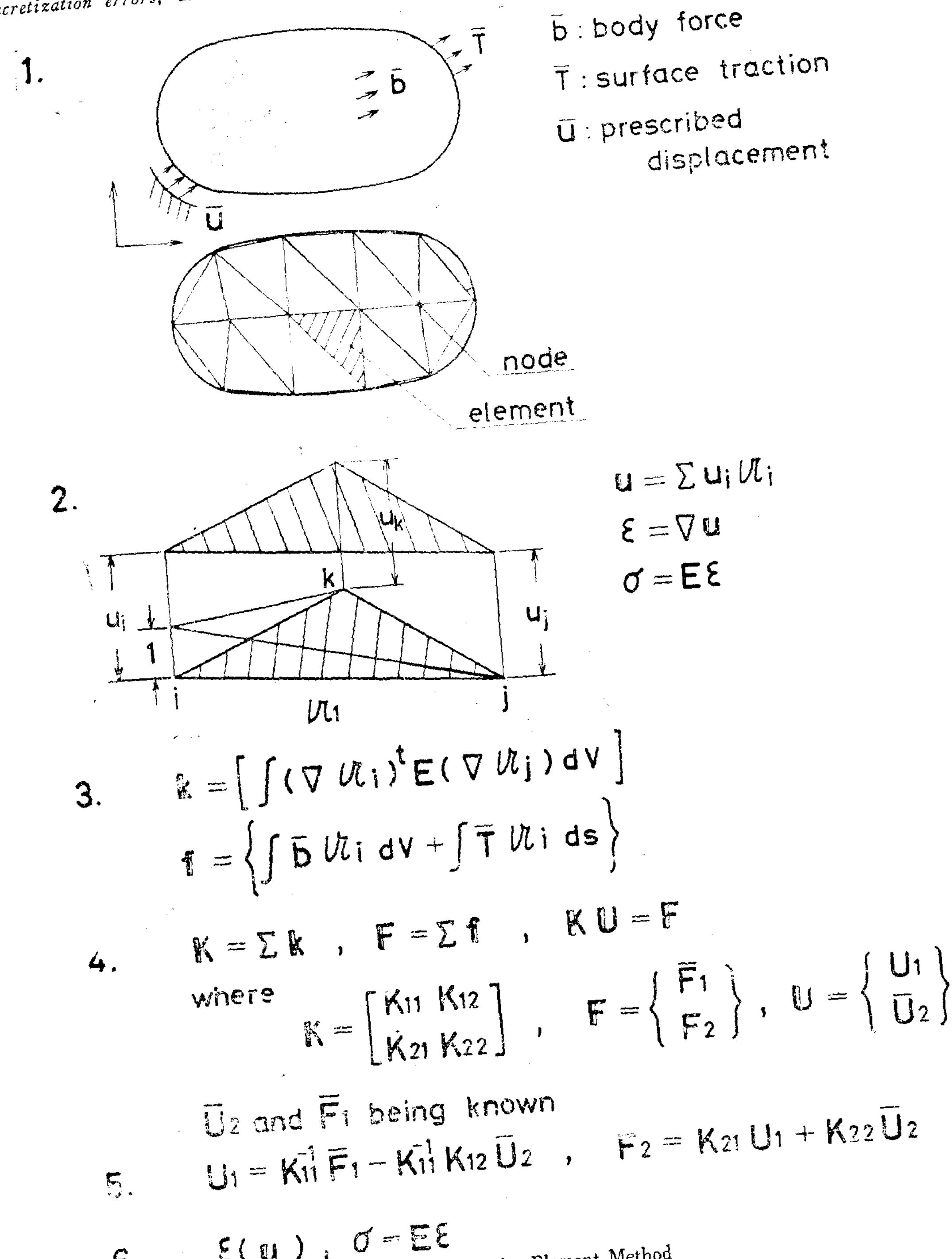


Fig. 1 Finite Element Method

caused in this process are called *rounding* or *round-off errors*, and in general, they increase with the number of nodal displacements.

Moreover, it should be noticed that the economical viewpoint is an important feature of the finite element method, because it computaions may be very expensive.

Program Development for Ship Structure Analysis[4]

Many computer programs for structural analysis have been developed, and large-scaled programs are categorized into two; *general purpose pregram* and *special purpose program*. A general purpose program is capable to analyze many kinds of problems, but its application to smaller problems may be ineconomical in general. ASKA, NASTRAN and SESAM 67 are the most famous ones of this kind.[4,5,6] A program consists of four parts:

1. Input.
2. Element Library.
3. Solution.
4. Output.

The number of unknowns treated by a large-scaled program may be hundreds of thousands, and therefore, *solution* should be carefully designed. For this purpose, the multi-layered substructure method is effective. A huge number of input and output data are handled for this kind of programs, and it is really laborious work. Special purpose programs are designed to save labor for preparing input data and visualizing output data by storing appropriate subprograms called *input* and *output generators* in conformity to the problems of a special kind. PASSAGE is designed as a special purpose program for tankers, ore carriers, and bulk carriers.[8] A general purpose program can be regarded as a special purpose program if appropriate input and output generators are attached. Input and output generators for NASTRAN were developed by Lloyd Register of Shipping. A special purpose program is convenient to analyze problems of special kind; their

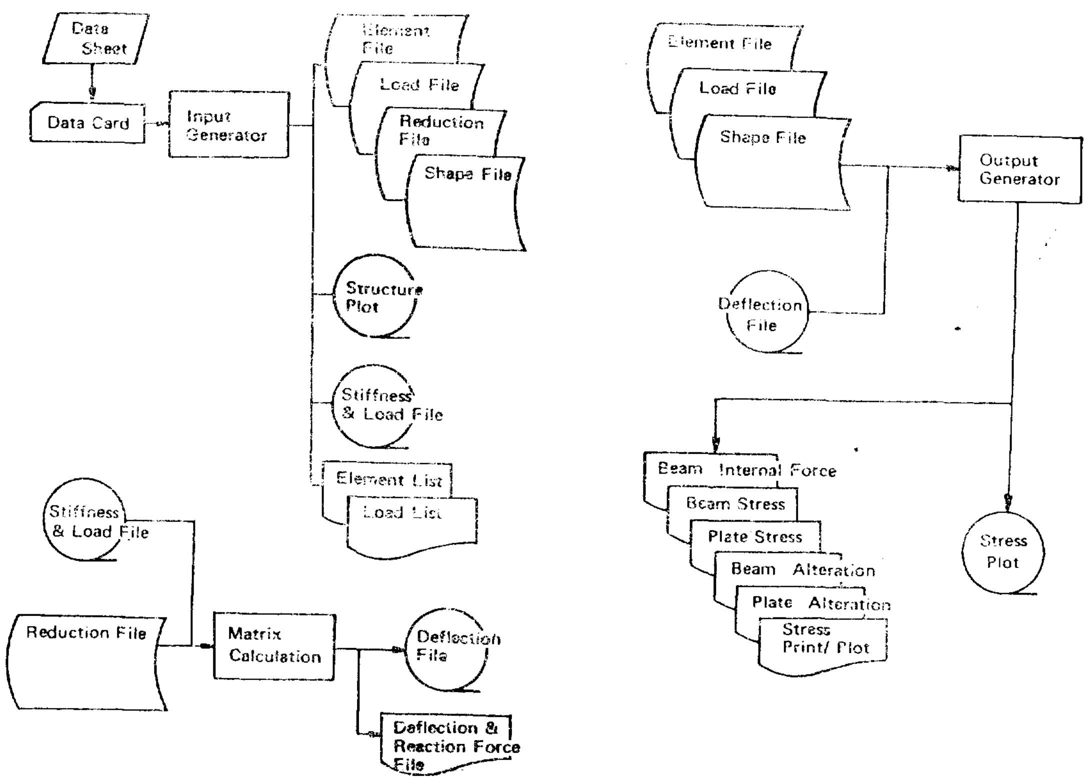


Fig. 2 Program Bcstap

input and output generators are, however, extremely specialized and cannot be applied to structures of other classes.

There are so many programs of moderate or small size. STRUDL and SAP programs are general purpose programs of moderate size, and can be conveniently applied to local strength analysis of ships.[9] A special purpose program named BCSTAP was developed by the present author for transverse strength analysis of bulk carriers;[10] it is designed for dailyworks at Nippon Kaiji Kyokai, and its general feature can be seen in Fig. 2. Input data for BCSTAP can be prepared in a half day by one man. For local strength analysis, small-scaled programs for plane stress problems can be conveniently applied.

Transverse strength of a ship can be analyzed by the frame analysis method, which can be considered as a kind of the finite element method, and it is very effective and economical for special types of ships like tankers. As an important feature, sectional forces and moments are calculated directly by this analysis together with stress values, and they are most familiar quantities for designers. It should be noticed that there is a limitation for the range of application by this method; it is inadequate for double bottom analysis.

Stress Concentrations and Stress Intensity Factors

Stress distributions in ships can be roughly estimated by the conventional beam theory or the three-dimensional finite element method. Detailed stress distributions around a rounded part can be calculated by the finite element method by finely subdividing the structural part under consideration. For this purpose, the substructure technique and the zooming technique can effectively applied; the latter is a special instance of the former, and is much more convenient for practical applications. In the case of the zooming technique, finite element solutions with fine mesh subdivision are obtained only for a restricted domain

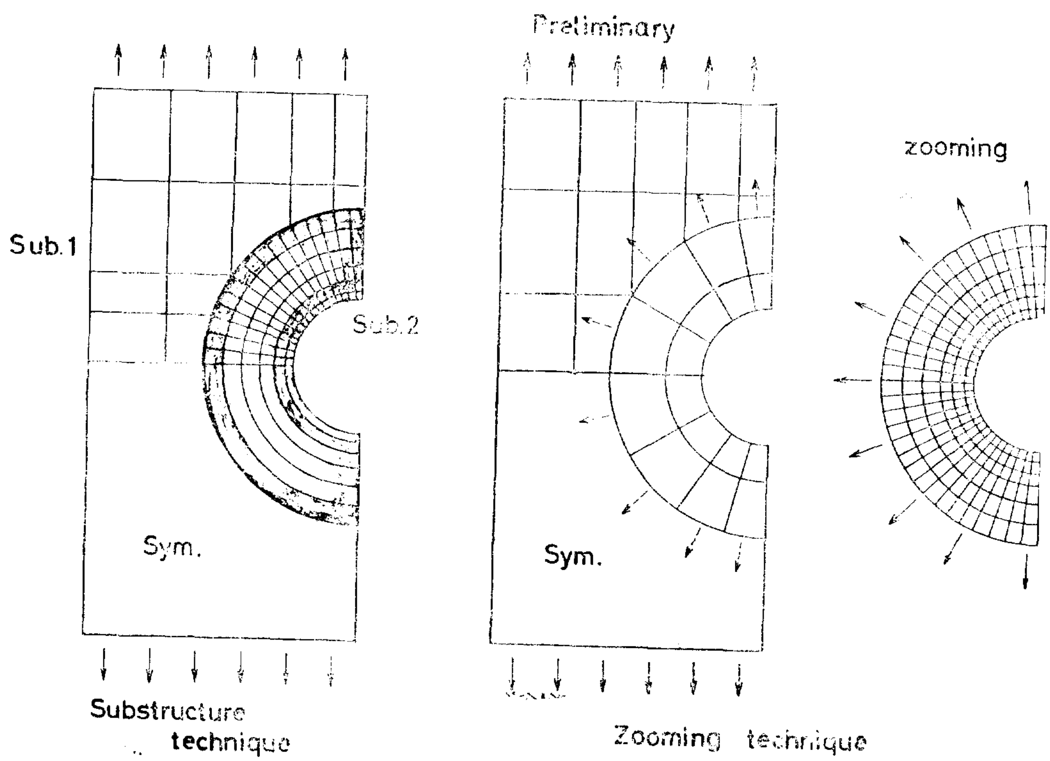
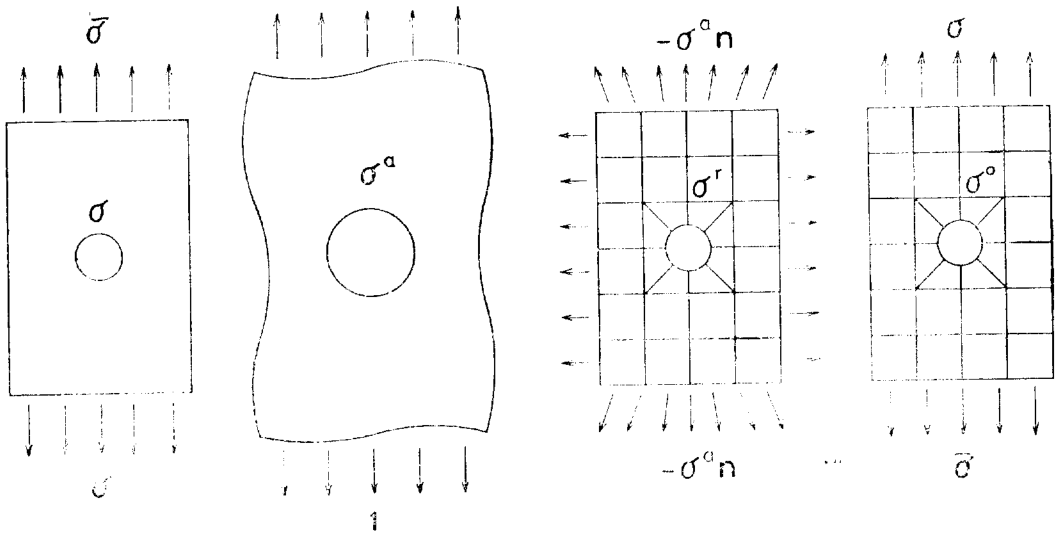


Fig. 3. Substructure technique and zooming technique



$$\sigma = c(\sigma^a + \sigma^r) + \sigma^o$$

Fig. 4. Method based on superposition

(Fig. 3) which covers the part under consideration, and are in equilibrium on the boundary of this domain with the stresses obtained by the preliminary calculations. It should be noticed that precision of solutions obtained depends largely upon the size of the domain for zooming analysis; the size should be large in comparison to that of the rounded part.

The author proposed a special technique for analyzing stress concentrations on the basis of the concept of superposition of analytical and finite-element solutions.[11] In most cases, the stress distribution σ around a cutout can be estimated with an analytical solution σ^a within an unknown parameter c; the residual part of solution can be determined without using fine mesh subdivision. The solution σ is given in the form

$$\sigma = c(\sigma^a + \sigma^r) + \sigma^o \qquad (1)$$

where σ^r and σ^o are the residual solutions obtained by the finite element method (Fig. 4). The unknown parameter c can be determined by introducing the above expression into the principle of minimum potential energy. The parameter c can also be determined accurately by the following simple condition:

$$c\sigma^r + \sigma^o = 0 \qquad (2)$$

where the stresses are evaluated for the maximum stress component at the point of stress concentration. This condition can be derived from the principle of minimum potential energy by omitting small quantities. This idea can be applied widely to various problems.

Stress distributions around a crack front are characterized by the stress intensity factor, which is closely related to fatigue and brittle fracture and plays a central role in linear fracture mechanics. Consider a cracked plate stressed symmetrically with respect to the crack plane (cf. Fig. 5). The stress near the crack front is given by

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau \end{Bmatrix} = \frac{K}{\sqrt{2\pi r}} \begin{Bmatrix} \cos(\theta/2)[1-\sin(\theta/2)\sin(3\theta/2)] \\ \cos(\theta/2)[1+\sin(\theta/2)\sin(3\theta/2)] \\ \sin(\theta/2)\cos(\theta/2)\cos(3\theta/2) \end{Bmatrix} + O(1) \qquad (3)$$

where K is the stress intensity factor. A large number of methods for determining it by using the finite element method have been proposed, and the simplest method is the use of crack opening displacement near

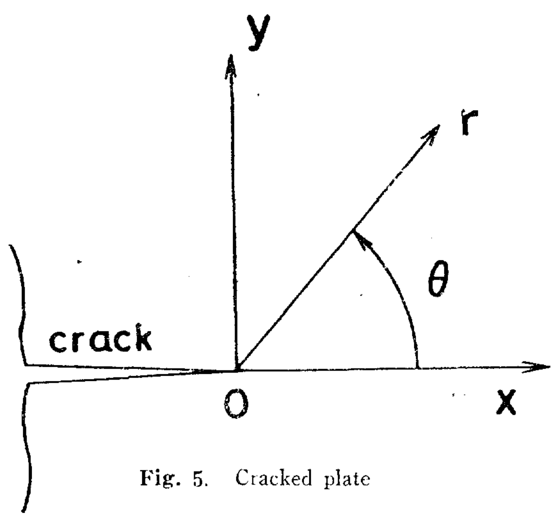


Fig. 5. Cracked plate

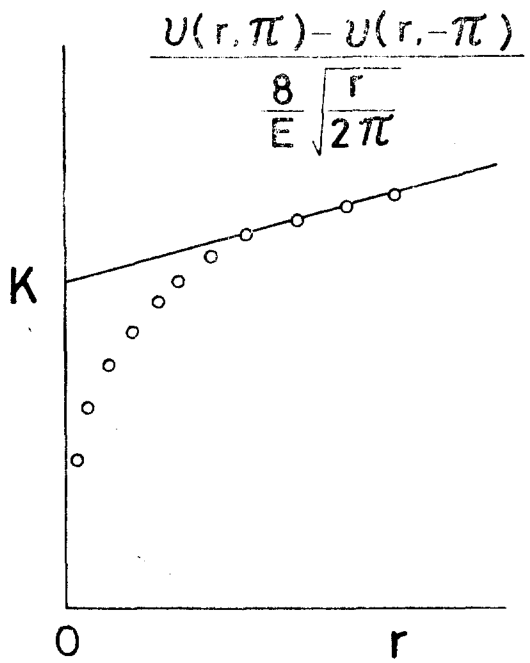


Fig. 6. Stress intensity factor

the crack front. The difference of displacements of a pair of points$(r, \pm\pi)$ on the crack plane near the crack front is given in the form

$$v(r, +\pi)-v(r, -\pi)=\frac{8}{E}\left(\frac{r}{2\pi}\right)^{1/2}(K+Dr) \quad (4)$$

where E is Young's modulus. Plotting numerical results for the right hand side of Eq. (4) versus r gives the stress intensity factor K(cf. Fig. 6).

The stress intensity factor has a similar property to the stress concentration factor, and therefore, the method based on the concept of superposition of analytical and finite-element solutions can be applied effectively.[12] The first term on the right hand side of Eq. (3) can be regarded as the analytical stress distribution, if K is considered as an unknown parameter. The condition given by Eq. (2) takes the following form:

$$K\ \sigma_y^r+\sigma_y^o=0 \quad (2')$$

where the left hand side is evaluated at the crack front, and σ_y^r and σ_y^o are the y-components of σ^r and σ^o which can be determined as before. The parameter K determined by this condition is the stress intensity factor. The present method can be applied to complicated problems including three-dimensional ones.

Numerical Simulation

The finite element method can be applied not only to structural analysis but also to simulation of processes, such as cutting and welding of plates and loading and unloading of cargoes on board. Here behaviors of ores in an ore carrier will be dealt with, and is regarded as an interaction problem of a ship-cargo system.[13] The ore cargo behaves as a structural element in an ore carrier, and shows nonlinear elasticity and yielding as a continuum. The elasticity is governed by the following complimentary energy per unit volume:

$$U_c=C_1J_1^2+C_2J_2+C_3(J_2/J_1)^2\ (J_1<0) \quad (5)$$

where J_1 and J_2 are the first and second stress invariants, and C_1, C_2 and C_3 are the positive numerical constants corresponding to elastic coefficients. The third term in the expression for U_c governs the phenomenon called *dilatancy*. As for yielding, ores follow the Drucker and Prager criterion given by

$$f\equiv\alpha J_1+J_2^{1/2}-k=0 \quad (6)$$

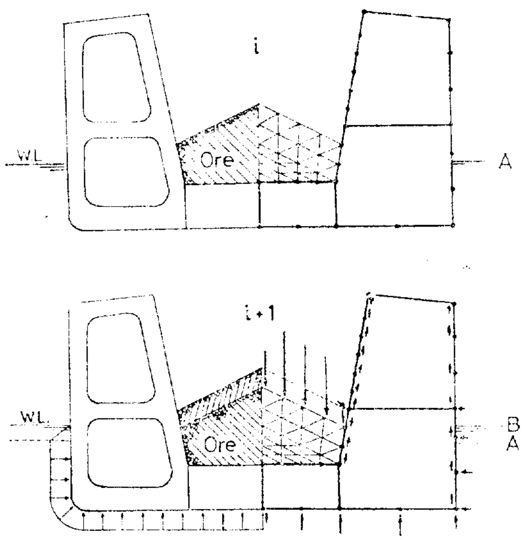
where α and k are constants. For Kaiser ore pellets, it can be assumed on the basis of experiments that $C_1=0.03$, $C_2=0.25$, $C_3=1.5(\text{mm}^2/\text{kg})$, $\alpha=0.27$, $k=0$

As can easily be seen, the mechanical property of ores depends upon the instantaneous stress which is closely related to loading pathes of ores.

The ore stress distribution just after loading has been estimated by the Coulomb theory which can be regarded as a rigid-plastic analysis method. Here it will be done by the finite element method according the numerical simulation procedure which has been employed for excavations and embankments in soil and rock mechanics.[2] The hull structure and the ore field are subdivided into finite elements in conformity with the sequence of ore loading. Consider the loading stages i and $i+1$ shown in Fig. 7. The change of the loading stage from i to $i+1$ is realized by adding a layer of ore and by changing the water line from A to B. Let σ_i and σ_{i+1} be the stresses at these two stages. Then the increment

$$\Delta\sigma=\sigma_{i+1}-\sigma_i \tag{7}$$

can be obtained with the aid of the finite element analysis by considering the ore field and the hull as one structural system and the stress σ_{i+1} thus determined will be used for the analysis at the next stage. Fig. 8 shows the Fushu-maru on which Hagiwara and Tani measured ore pressure on longitudinal bulkheads; the ore field is idealized by triangular elements, and the hull by beam elements. The ore treated here is iron ore fines of specific weight 2.6×10^{-3} kg/cm^3. The ore pressure distribution just after loading can be calculated by the present procedure, and the results are shown in Fig. 9 together with Hagiwara's experiments, which shows fairly good agreements.



Fi·. 7. Loading procedure and element subdivision

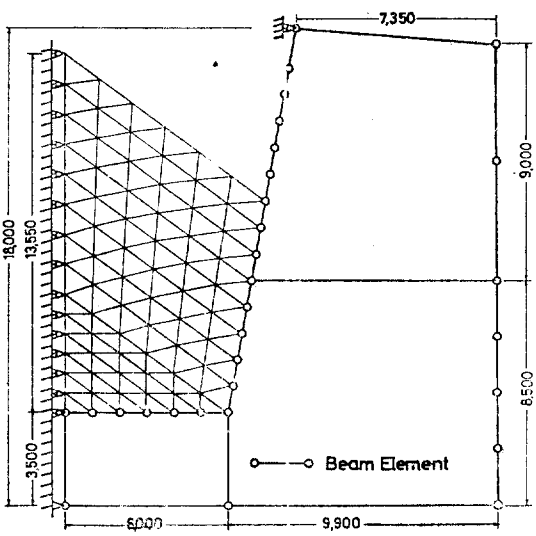


Fig. 8. Fushu-maru

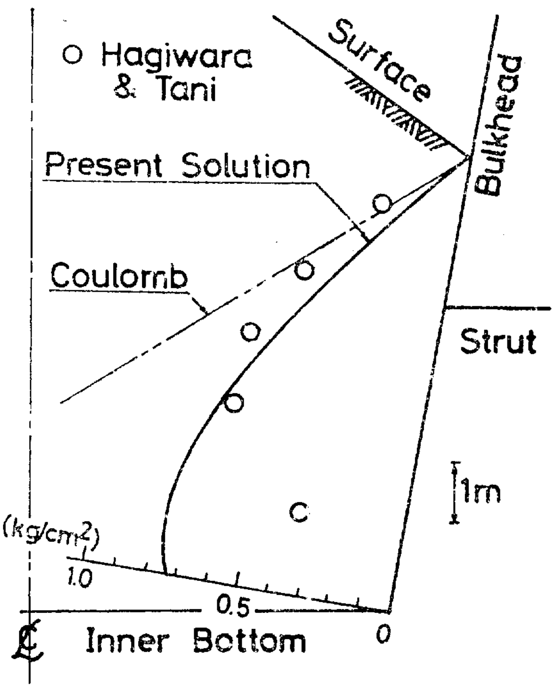


Fig. 9. Ore pressure distributions on longitudinal bulkhead

Buckling and Postbuckling

Buckling of a plate shown in Fig. 10 can be analyzed by introducing the so-called geometrical stiffness matrices, which corresponds to the terms including

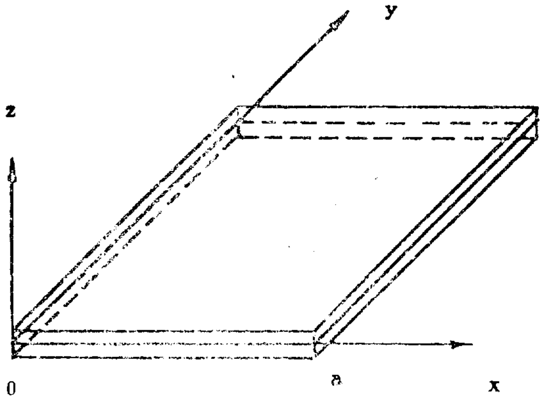


Fig. 10. Rectangular plate

instantaneous stresses in the linearized equation of equilibrium given by

$$D\Delta\Delta w = t\left(\sigma_x \frac{\partial^2 w}{\partial x^2} + 2\tau \frac{\partial^2 w}{\partial x \partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2}\right) \qquad (8)$$

where w is the displacement in the z-direction, and t and D are the thickness and the flexural rigidity of the plate. In most cases, ships' structural elements in compression have a rectangular form, and the deflection w can be expressed in a Fourier series; if the plate is simply supported at the transverse edges, the deflection is given exactly in the form

$$w(x, y) = \sum_{n=1}^{\infty} W_n(y) \sin\frac{n\pi x}{a} \qquad (9)$$

where a is the length of the plate in the direction of compression, and W_n's are functions of y alone. The coefficient W_n can be discretized in the y-direction by the same way as the conventional one-dimensional finite element method, and is expressed in terms of the values at nodes or nodal lines. Moreover, the expression given by Eq. (9) is approximated by a finite Fourier series as

$$w(x, y) = \sum_{n=1}^{N} W_n(y) \sin\frac{n\pi x}{a} \qquad (9')$$

The finite element formulation is easily be derived by the Galerkin method; multiplying both sides of Eq. (8) by

$$\delta w(x, y) = \Sigma\ \delta w_n(y) \sin\frac{n\pi x}{a} \qquad (10)$$

and integrating the resulting relation throughout the mid-plane of the plate lead to the final equation. The finite element method of this type is often called the finite strip method.

By this procedure, a two-dimensional problem can be converted into a set of one-dimensional problems. The finite strip method can be applied not only to buckling analysis but also to postbuckling problems. In the case of elasto-plastic deformations, this technique has a tendency to loose accuracy from the theoretical point of view. Fig. 11 shows a complicated plate structure in compression, which can be analyzed by the use of strip elements and conventional beam elements. The results are shown in Fig. 12 together with experimental results, which shows fairly good agreements.

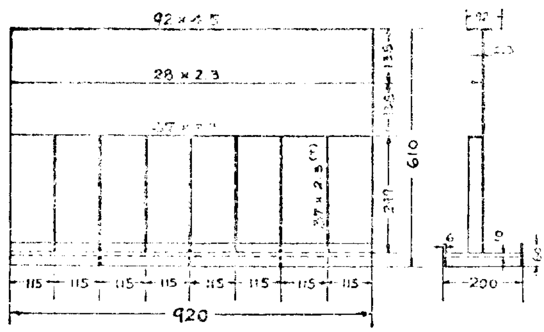


Fig. 11. Bottom transverse

For determining buckling loads, an eigenvalue problem should be solved, and it is, in general, given as a large-scaled problem, although only the lowest eigenvalue is to be obtained. For this kind of problems, the Rayleigh-Ritz method has been commonly used. An extension of this method, called the simultaneous iteration method or the subspace method, can be efficiently used for this problem. Assume the eigenvalue problem is given in the form

$$Ax = \kappa Bx \qquad (11)$$

where A and B are positive-definite symmetric matrices, and κ is eigenvalue.[14,16] Both of A and B are not necessarily positive-definite in practical problems, but they can be rewritten in the above form by an appropriate transformation.

The present method starts with chosing a set of starting iteration vectors x_i^0 $(i=1,\cdots\cdots,p)$. Algorithm of this method is as follows;

1. $X_{(0)} = [x_1^0, \cdots\cdots, x_p^0]$
   $Z_{(0)} = A\ X_{(0)}$
2. go to 4;

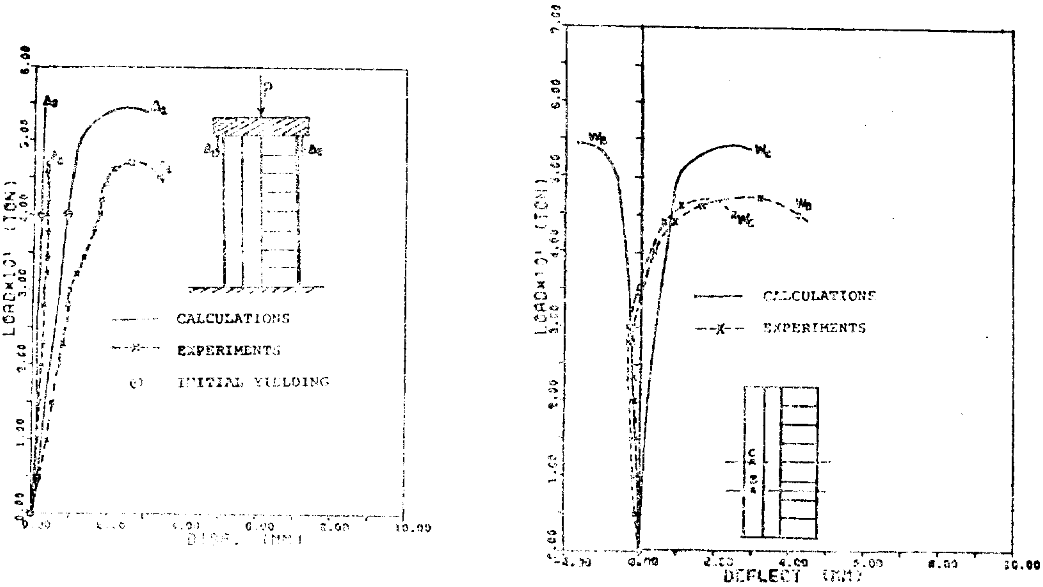


Fig. 12. Comparison of theory and experiment for bottom transverse in compression

3. solve $AX_{(k)}=Z_{(k)}$
4. $U_{(k)}=BX_{(k)}$
5. the Cholesky factorization
   $X_{(k)}{}^tZ_{(k)}=R_{(k)}{}^tR_{(k)}$
   where $R_{(k)}$ is an upper triangular matrix, and t indicates the transpose of an appropriate quantity;
6. inversion $\bar{R}_{(k)}=R^{-1}{}_{(k)}$
7. $\bar{B}_{(k)}=(X_{(k)}\bar{R}_{(k)})^tU_{(k)}\bar{R}_{(k)}$
8. solve the eigenvalue problem
   $\bar{B}_{(k)}\ Q_{(k)}=Q_{(k)}\ \Lambda_{(k)}$
   where $Q_{(k)}{}^tQ_{(k)}=I$=unit matrix, and $\Lambda_{(k)}$=diag $(1/\kappa_{(k)1},\cdots\cdots,1/\kappa_{(k)p})$
9. convergence test;
10. if solutions are convergent, put
    $Z_{(k+1)}=U_{(k)}Q_{(k)}$
11. return to 3.

In engineering problems, A and B are given as band matrices, and this algorithm takes advantage of this property, and can be applied efficiently on virtual-memory computers.

Elimination of unwanted variables is often used for the same purpose.[18] It can be regarded as an special instance of the Rayleigh-Ritz procedure, and may give significant errors if variables to be eliminated are chosen improperly.

If once the eigenfunctions or the buckling modes are determined, postbuckling behaviors can be analyzed by the Ritz procedure by using the buckling modes as the coordinate functions. The finite element equations can be solved directly, but it is generally inefficient for practical purpose. It should be noticed that finite elements other than the socalled conforming elements may give significant errors in buckling and post buckling problems.

Moreover, a free vibration problem is given as an eigenvalue problem, and the above methods for eigenvalue problems can be applied to it.

Conclusions

The present paper deals with basic applications of the finite element method to structural problems; the fundamental theories for them are almost established. Structural optimization and dynamic problems are not treated herein, though they are getting more important. For these problems, efficient optimization techniques or time integration schemes play important roles.[17]

The author wishes to express his sincere gratitude to Professor S. J. Yim, President of the Society, for his kind invitation.

References

1. O.C. Zienkiewicz, *The Finite Element Method in Engineering Sciences*, McGraw-Hill, 1967.
2. C.S. Desai & J. F. Abel, *Introduction to the Finite Element Method*, Van Nostrand, 1972.
3. G. Strang & G. J. Fix, *An Analysis of the Finite Elements Method*, Prentice Hall, 1973.
4. W. Pilkey, K. Saczalski, & H. Schaeffer (ed.), *Structural Mechanics Computer Program*, Univ. Press of Virginia, 1974.
5. C.W. McCormick, Sparse Matrix Operations in NASTRAN, *Theory and Practice in Finite Element Structural Analysis*, Univ. of Tokyo Press, 1973.
6. G. von Fuchs & E. Schrem, ASKA-A Computer System for Structural Engineers, *Finite Element Techniques*, ISD Univ. of Stuttgart, 1971.
7. E.M.Q. Rren, Impact of Finite Element Techniques on Practical Design of Ship Structures, *ibid.*
8. M. Yoshiki, T. Okabe, T. Kawai, & M. Hidaka, On the Development of the "PASSAGE" Program, *Theory and Practice in Finite Element Structural Analysis*, Univ. of Tokyo Press, 1973.
9. E.L. Wilson, SAP-A General Structural Analysis Program for Linear Systems, *Advances in Computational Methods in Structural Mechanics and Design*, UAH Press, Univ. of Alabama in Huntsville, 1972.
10. Y. Yamamoto, Y. Masuda, S. Tashiro, & K. Inoue, A Program System for Structural Analysis of a Bulk Carrier, *Japan Shipuilding & Marine Engng.*, 1974.
11. Y. Yamamoto, Finite Element Approaches with the Aid of Analytical Solutions, *Recent Advances on Matrix Meth. of Structural Analysis and Design*, 1969.
12. Y. Yamamoto, N. Tokuda, & Y. Sumi, Finite Element Treatment of Singularities of Boundary Value Problems and its Application to Analysis of Stress Intensity Factors, *Theory and Practice of Finite Element Meth. for Struc. Analysis*, Univ. of Tokyo Press, 1973.
13. Y. Yamamoto & T. Ura, Distribution of Ore Pressure in an Ore Carrier, *Int. Shipbuilding Progress*, 1977.
14. Y. Yamamoto & H. Ohtsubo, Subspace Iteration Accelerated by Using Chebyshev Polynomials for Eigenvalue Problems with Symmetric Matrices, *Int. J. Num. Meth. in Engng.*, 1976.
15. B.R. Corr & Jennings, Implementation of Simultaneous iteration for vibration analysis, *Computers & Structures*, 1973.
16. K.J. Bathe & E. L. Wilson, Solution Methods for Eigenvalue Problems in Structural Mechanics, *Int J. Num. Meth. Engng.*, 1973.
17. *Numerical Methods*, Report of Committee IV. 1, ISSC 1976.
18. B. Irons, Structural Eigenvalue Problems: Elimination of Unwanted Variables, *AIAA J.*, 1965.