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ALMOST WEAKLY-OPEN Π-IMAGES OF METRIC SPACES

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The probem of determining which properties of a class of topological spaces are preserved under various mappings is fundamental to general topology, and one specific problem is to characterize classes of spaces under a particular map (Throughout this paper, maps are considered to be continuous and spaces to be Hausdorff). V. I. Ponomarev[6] proved that a space is developable if and only if it is an open Π-image of a metric space. A. V. Arhangel'skii[3] showed that a quotient Π-image of a metric space is symmetrizable via a symmetric satisfying the weak condition of Cauchy, while Ja. A. Kofner[4] proved the converse. Considering these two cases, we can guess that a natural candidate for g-developable spaces must be more restrictive than quotient Π-maps and more generalized than open Π-maps. In fact, what we will prove in this paper is: A space is g-developable if and only if it is an almost weakly-open Π-image of a metric space.

A space X is *g-developable* [5] if there is a map g on $N\times X$ to 2^X such that $\mathcal{W}_x=\{g(n,x): n\in N\}$ forms a weak-base (in the sense of [2]) and that a sequence $\{x_n\}$ converges to x, if $x, x_n\in g(n, y_n)$ for some y_n for each $n\in N$. Such a map g is called a g-developable CWC-map (=countable weakly-open covering map). K. B. Lee[5] characterized such spaces as: A Hausdorff space is g-developable if and only if it is symmetrizable via a symmetric under which all convergent sequences are Cauchy.

A mapping f from a metric space (X,d) onto a space Y is called a *Π-map*[6] if for each $y\in Y$ and each neighborhood V of y,

$$d(f^{-1}y,\ X\text{-}f^{-1}V)>0.$$

A map $f:X\longrightarrow Y$ is said to be *almost open*[1] if for each $y\in Y$, there exist an $x\in f^{-1}y$ and a local base $\mathcal{B}_x$ at x such that $f(B)$ is open for each $B\in\mathcal{B}_x$. Some properties of almost open maps may be found in [1]. Now we introduce a more general concept.

DEFINITION. A map f: $X\longrightarrow Y$ is said to be *almost weakly-open* if for each $y\in Y$, there exist an $x\in f^{-1}y$ and a local base $\mathcal{B}_x$ at x such that $\mathcal{W}_y=\{f(B):B\in\mathcal{B}_x\}$ forms a weak-base.

Note that the condition "there exist an $x\in f^{-1}y$ and a local base $\mathcal{B}_x$ at x" may be exchanged by "there exists an $x\in f^{-1}y$ such that for any local base $\mathcal{B}_x$ of x". Also note that an open (of course, continuous) map is almost weakly-open and that an almost weakly-open (again, continuous) map is quotient. We are now at the position to state the main theorem.

THEOREM. *A space is g-developable if and only if it is analmost weakly-open Π-*

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image of a metric space.

Proof. Let f be an almost weakly-open Π-map from a metric space (X, d) onto a space Y. For each $x\in X$ and $n\in N$, let $g(n,x)=f(S(X'; 1/n))$, where x' is the specified point of $f^{-1}x$ and $S(x'; 1/n)$ is the $1/n$-sphere centered at x'. It can be easily verified that g is a g-first countable CWC-map [5].

Let $x, x_n\in g(n, y_n)$ for each $n\in N$. This implies

$$f^{-1}x_n\cap S(f^{-1}x;\ 2/n)\neq\phi.$$

To show that the sequence $\{x_n\}$ converges to x, let a neighborhood V of x be given. Since f is a Π-map, there exists an $\varepsilon>0$ such that $d(f^{-1}x,\ X-f^{-1}V)\geqq\varepsilon$, which implies $f(S(f^{-1}x;\varepsilon))\subset V$. If we take any $n\in N$ such that $2/n<\varepsilon$,

$$f^{-1}x_n\cap S(f^{-1}x;\varepsilon)\supset f^{-1}x_n\cap S(f^{-1}x;1/n)\neq\phi.$$

It follows that $x_n\in f(S(f^{-1}x;\varepsilon))\subset V$ for every $n\in N$ such that $2/n<\varepsilon$, that is, the sequence $\{x_n\}$ is eventually in V.

Conversely, let Y be a g-developable space with a g-developable CWC-map g. We will construct a metric space Z and an almost weakly-open Π-map f from Z onto Y. Let $Z=\{\{x_n\}:\{x_n\}$ is a sequence in Y such that $\bigcap_{n\in N} g(n,x_n)\neq\phi\}$. Now topologize Z by a metric

$$d(\{x_n\},\{y_n\})=\sqrt{\sum\left(\frac{1}{2}\right)^n(x_n-y_n)},$$

where $(x-y)=1$, if $x\neq y$; 0, otherwise. Define a map $f\colon Z\longrightarrow Y$ by $f(\{x_n\})=\bigcap_{n\in N} g(n,\ x_n)$. Note that if $\{x_n\}\in Z$, the intersection $\bigcap g(n,x_n)$ is singleton, and the sequence $\{x_n\}$ converges to that point. This shows that f is well defined. Each fiber has of the form

$$f^{-1}x=\{\{x_n\}\in Z\colon\ x\in\bigcap g(n,x_n)\}.$$

Given a neighborhood O_x of a point x of Y, there exists a $k\in N$ such that $g(k,y)\subset O_x$ if $x\in g(k,y)$. Therefore, for all $\{x_n\}\in f^{-1}x$, $g(k,x_k)\subset O_x$. On the other hand, if $\{y_n\}\in Z-f^{-1}O_x$, $g(n,y_n)$ intersects $Y-O_x$ for each $n\in N$. Now we have $d(f^{-1}x,\ Z-f^{-1}O_x)\geqq\sqrt{\frac{1}{2^k}}$, which completes the proof that f is a Π-map.

Next, we will show that f is almost weakly-open. For each $X\in Y$, let $<x>=\{x,x,x,\cdots\}\in f^{-1}x$ be a specified point of $f^{-1}x$. It suffices to show that $\mathfrak{B}_x=\{f(S(<x>; 1/n)):n\in N\}$ forms a weak-base. Let M be a subset of Y such that for each $y\in M$ there exists an $n\in N$ such that $f(S(<y>:1/n))\subset M$. For a fixed $y\in M$, let $f(S(<y>;1/m))\subset M$. Let k be a positive integer such that $\sqrt{\sum_{i>k}\frac{1}{2^i}}<\frac{1}{m}$. For any $z=\{y,y,\cdots,y,*,*,*,\cdots\}\in Z$, where the number of $y's$ is k and * is taken in Y arbitarily, $d(z,<y>)<\frac{1}{m}$. Then $f(z)\in f(S(<y>;\ 1/\mathrm{m}))\subset M$. This implies

$$\bigcap_{i\leq k} g(i, y) \cap \bigcap_{j>k} g(j, *) \subset M$$

for any choice of * if the intersection is nonempty. Assume $g(k, y) \not\subset M$. Choose an $x \in g(k, y) - M$. Then $x = f(\{y, y, \cdots, y, x, x, \cdots\}) \in M$, a contradiction. Thus, $g(k, y) \subset M$. This is true for any $y \in M$ so that M is open in Y. This completes the proof.

COROLLARY. *A space is symmetrizable via a symmetric under which all convergent sequences are Cauchy if and only if it is an almost weakly-open Π-image of a metric space.*

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