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ON CROSSINGS, RETURNS, POSITIVE RETURNS AND POSITIVE STEPS IN RANDOM WALK

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Introduction:

Chung and Feller (1949), Feller (1957), Csaki and Vincze (1961), Kanwar Sen (1964) and Jain (1966) have considered the problem of coin tossing game and random walk of $2n$ steps when final position is specified and when it is not by taking a sequence of independent random variables X_j, $j=1,2,\cdots$ each taking values ± 1 with probability $\frac{1}{2}$ and studied the behaviour of partial sums $S_n=\sum_{j=1}^{n} X_j$. Authors have earlier considered the problem of fluctuations in coin tossing and random walk of $2n$ steps starting from origin and terminating at $(2n, 0)$ satisfying the event 'E' for the sequence of partial sums S_n that: If $S_r=0$ for $r=2\alpha_1, 2\alpha_2, \cdots, 2\alpha_i=2n$; $i=1,2,\cdots, n$ then the j^{th} segment included between the two consecutive zeroes i.e. between the $(j-1)^{th}$ and j^{th} zero satisfy the condition:

$$E\begin{cases} 0=S_{2\alpha_{j-1}}<S_{2\alpha_{j-1}+1}<S_{2\alpha_{j-1}+2}<\cdots<S_{\alpha_j-\alpha_{j-1}}>S_{\alpha_j-\alpha_{j-1}+1}>\cdots>>S_{\alpha_{2j}}=0 \\ 0=S_{2\alpha}>S_{2\alpha_{j-1}+1}>S_{2\alpha_{j-1}+2}>\cdots>S_{\alpha_j-\alpha_{j-1}}<S_{\alpha_j-\alpha_{j-1}+1}<<\cdots<S_{\alpha_{2j}}=0 \end{cases}$$

for $j=1,2,\cdots i$ and $i=1,2,\cdots n$.
Alternatively E can be described as "the sequence $S_0, S_1\cdots S_n$ does not satisfy the condition: $S_{j-1}>S_j<S_{j+1}$ when $S_j>0$ and $S_{j-1}<S_j>S_{j+1}$ when $S_j<0$ for all j".

In this paper we consider the problem of symmetric random walk of $2n$ steps from $(0,0)$ to $(2n, 2m)$ satisfying event E upto the last return to the x-axis but the segment from this last return point to the terminus at $(2n, 2m)$ of length 2α $(m\leq\alpha\leq n)$ say, should be of the type that it first takes $\alpha+m$ positive steps and then come down to the terminus $(2n, 2m)$ after having $(\alpha-m)$ negative steps. Let this condition be called (A) Such a path from $(0,0)$ to $(2n, 2m)$ satisfying event 'E' and (A) will be said to satisfy event 'F'.

In random walk terminology, it may be interpreted as: Starting from x-axis, it moves up (down) as many steps as it likes but once it changes the direction, it will continuously move downwards (upwards) until it comes to the x-axis. In the next step it way go up or come down and repeats the above pattern i.e. if it is moving upwards (downwards) it goes on moving but once it moves downwards (upwards) it will continuously move down (up) until it again comes to the x-axis. In other words, "particle approaching the starting level does not change its direction until it reaches there." After the last re-

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turn and upto the terminus $(2n, 2m)$, particle follows the path satisfying (A).

We obtain then the probability that a particle starting from the origin and reaching $(2n, 2m)$ at the $2n^{th}$ step satisfying event 'F' and having

(1) r returns to the x-axis.

(2) r_1 positive returns to the x-axis.

(3) b crossing with the x-axis

(4) r returns, r_1 positive returns and b-crossings with x-axis

(5) r returns, r_1 positive returns, b crossings and $2h$ steps above x-axis.

Path Representation:

Let X_j be a random variable associated with the j^{th} step of the simple random walk, taking two values ± 1 according to as the particle has a positive and negative step respectively. Writing $S_0=0$, $S_j=X_1+X_2+\cdots+X_j$ $(j>0)$, the $S_0, S_1, \cdots S_n$ satisfy the condition $S_j-S_{j-1}=X_j=\pm 1$, $j=1, 2, \cdots n$. Using the geometrical terminology the points (j, S_j) when plotted on a x-y plane and joined successively by straight line segments, we get a path whose vertices have abscissa $0, 1, \cdots, n$ and ordinates $S_0, S_1, \cdots, S_n$ respectively. Such a path may be taken as representing the simple random walk.

Notations:

A-point	: a point (j, S_j) with $S_j=0$ i.e. a return to the x-axis.
$A^+(A^-)$	: an A-point s.t. $S_{j-1}=+1$ $(S_{j-1}=-1)$. It is a positive (negative) return point.
V(wave)	: a segment of a path included between two consecutive A-points. The segment from origin to the first return point is also regarded as a wave.
$V^+(V^-)$	: a wave (V) with $S_j>0$ $(S_j<0)$ at the intervening position.
B (crossing or intersection with x-axis)	: a point (j, S_j) of the path with $S_j=0$ and $S_{j-1}\cdot S_{j+1}=-1$.
C(section)	: a segment of a path included between two consecutive B-points. The segments from origin to the first B-point and that from the last B-point to the end point if on the x-axis are also regarded as sections.
$C^+(C^-)$	: a section C with $S_j>0$ $(S_j<0)$ in between.
$C_{n,m}$	: a path $S_0, S_1, \cdots, S_n$ with $S_n=m$, $0\leq m\leq n$ $C_{n,0}=C_n=0$ with n even.
$C^b_{n,m}$	: a $C_{n,m}$ with b B-points.
$C^b_{n,m}(+)$ $(C^b_{n,m}(-))$	: a $C^b_{n,m}$ with $S_1=+1(S_1=-1)$
$C_{n,m,r}$	: a $C_{n,m}$ with r A-points.
$C_{n,m,\cdot,r}$	: a $C_{n,m}$ with r_1 A^+-points.
$C^b_{n,m,r}$	: a $C_{n,m,r}$ with b B-points.
$C^b_{n,m,\cdot,r_1}$	: a $C_{n,m,\cdot,r_1}$ with b B-points.
C_{n,m,r,r_1}	: A a $C_{n,m}$ with r A-points, r_1 A^+ points
C^b_{n,m,r,r_1}	: a $C^b_{n\ m}$ with r A-points and r_1 A^+-points.

$C^{(h)b}{}_{n,m}$	: a $C^b{}_{n,m}$ with h steps above x-axis.
$C^{(h)}{}_{n,m,r}$	: a $C_{n,m,r}$ with h steps above x-axis.
$C^{(h)}{}_{n,m,\cdot,r_1}$	: a $C_{n,m,\cdot,r_1}$ with h steps above x-axis.
$C^{(h)}{}_{n,m,r,r_1}$	: a $C_{n,m,r,r}$ with h steps above x-axis.
$C^{(h)b}{}_{n,m,r,r_1}$	: a $C^b{}_{n,m,r,r_1}$ with h steps above x-axis.
T^+(tail)	: segment of a path starting from x-axis and ending with a positive position but not touching the x-axis in between i. e. for a positive tail between the j^{th} step and n^{th} step, $S_j=0$, $S_i=0$, $i=j+1$, $j+2$, $\cdots$, n.
$(\cdots)F$	: No. of paths of the type$\cdots$which all satisfy the event F.
$(\cdots)^o{}_F$	: $(\cdots)_F$with even number of crossings.
$(\cdots)^e{}_F$	: $(\cdots)_F$ with odd number of crossings.

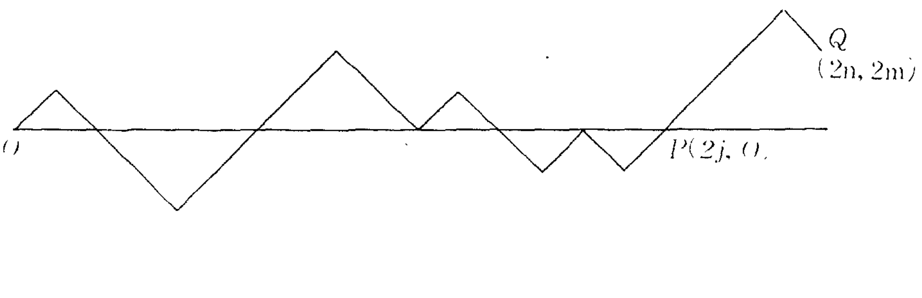
Weshall use the following results frequently in the sequel.

$$\left.\begin{array}{ll}(\mathrm{I}) & \sum_{j=0}^{n}\binom{a}{j}\binom{b}{n-j}=\binom{a+b}{n} \\ (\mathrm{II}) & \sum_{j=0}^{k}\binom{a+k-j-1}{k-j}\binom{b+j-1}{j}=\binom{a+b+k-1}{k}\end{array}\right\}\ \text{(Feller [3])}$$

Theorem:

(1) $$\left(C_{2n,\,2m,\,r,\,r_1}^{(2h)2b}\right)_F=\binom{r_1}{b}\binom{r-r_1-1}{b-1}\binom{n-h-1}{r-r_1-1}\binom{h-m}{r_1}$$

(2) $$\left(C_{2n,\,2m,\,r,\,r_1}^{(2h)2b-1}\right)_F=\binom{r_1}{b-1}\binom{r-r_1-1}{b-1}\binom{n-b-1}{r-r_1-1}\binom{h-m}{r_1}$$



Let OPQ be a $C_{2n,\,2m,\,r,\,r_1}^{(2h)2b}$ path with $S_1=+1$ as envisaged in (1), $P(2j,0)$ being the last return point to the x-axis and PQ satisfies the condition (A). $S_1>0$ and $S_{2n-1}>0$. Two cases arise:

(i) when last crossing is a last return as well.

(ii) when last crossing is not a last return.

In (i) path would consist of bC^+ which can be constructed from r_1V^+ of length $(2h-2n+2j)$ steps, $(r-r_1)V^-$ of length $(2n-2h)$ steps giving rise to bC^- and a T^+ at the end of length $(2n-2j)$ steps. r_1V^+ of length $(2h-2n+2j)$ steps can be constructed in $\binom{h-n+j-1}{r_1-1}$ ways since each V^+ has to be of length at least 2 sptes and $\binom{r_1-1}{b-1}$ is the number of ways of constructing bC^+ out of r_1V^+. This is akin to distributing g similar balls into C distinct cells which is possible in $\binom{g-1}{c-1}$ ways. Therefore bC^+ can be constructed from r_1V^+ of length $(2h-2n+2j)$ steps in $\binom{h-n+j-1}{r_1-1}\binom{r_1-1}{b-1}$ ways. Similarly $\binom{n-h-1}{r-r_1-1}\binom{r-r_1-1}{b-1}$ is the number of ways of forming bC^- out of $(r-r_1)V^-$ of length $(2n-2h)$ steps. Thus joining the waves in order we get the number of paths when last crossingis a last return also

$$=\sum_{j=r_1}^{n-m}\binom{r_1-1}{b-1}\binom{h-n+j-1}{r_1-1}\binom{r-r_1-1}{b-1}\binom{n-h-1}{r-r_1-1}$$

$$=\binom{r_1-1}{b-1}\binom{r-r_1-1}{b-1}\binom{n-h-1}{r-r_1-1}\binom{h-m}{r} \quad \text{(Feller [3])}$$

Similarly in (ii) when last crossing is not a last return the path consists of $(b+1)$ C^+, bC^- which can be constructed from r_1V^+, $(r-r_1)V^-$ of lengths $(2h-2n+2j)$ steps and $2n-2h$ steps respectively. Proceeding as in (1) no. of paths

$$=\binom{r_1-1}{b}\binom{r-r_1-1}{b-1}\binom{n-h-1}{r-r_1-1}\binom{h-m}{r_1}$$

Adding the two numbers

$$\left(C^{(2h)2b}_{2n,\,2m,\,r,\,r_1}\right)_F=\binom{r_1}{b}\binom{r-r_1-1}{b-1}\binom{n-h-1}{r-r_1-1}\binom{h-m}{r_1}$$

which leads to (1)
Similarly (2) can be easily obtained.

Deductions:

(i) Summing (1), (2) over $b+r_1\le r\le n-h+r_1$ we get

$$\left(C^{(2h)2b}_{2n,\,2m,\,.\,.\,,\,r_1}\right)_F=\sum_{r=b+r_1}^{n-h+r_1}\binom{r_1}{b}\binom{r-r_1-1}{b-1}\binom{n-h-1}{r-r_1-1}\binom{h-m}{r_1}$$

$$=\binom{r_1}{b}\binom{h-m}{r_1}\binom{n-h-1}{b-1}\sum_{r=b+r_1}^{n-h-r_1}\binom{n-h-b}{r-r_1-b}$$

(3) $$\left(C^{(2h)2b}_{2n,\,2m,\,.\,.\,,\,r_1}\right)_F=2^{n-h-b}\binom{r_1}{b}\binom{h-m}{r_1}\binom{n-h-1}{b-1}$$

Similarly

(4) $$\left(C^{(2h)2b-1}_{2n,\,2m,\,.\,,\,r_1}\right)_F=2^{n-h-b}\binom{r_1}{b-1}\binom{h-m}{r_1}\binom{n-h-1}{b-1}$$

(ii) Summing (1) over $b\le r_1\le r-b$ and (2) over $(b-1)\le r_1\le r-b$ respectively

$$\left(C^{(2h)2b}_{2n,\,2m,\,r}\right)_F=\sum_{r_1=b}^{r-b}\binom{r_1}{b}\binom{h-m}{r_1}\binom{r-r_1-1}{b-1}\binom{n-h-1}{r-r_1-1}$$

$$=\binom{h-m}{b}\binom{n-h-1}{b-1}\sum_{r_1=b}^{r-b}\binom{h-m-b}{r_1-b}\binom{n-h-b}{r-r_1-b}$$

(5) $$\left(C^{(2h)2b}_{2n,\,2m,\,r}\right)_F=\binom{h-m}{b}\binom{n-h-1}{b-1}\binom{n-m-2b}{r-2b}\quad\text{(using I)}$$

(6) $$\left(C^{(2h)2b-1}_{2n,\,2m,\,r}\right)_F=\binom{h-m}{b-1}\binom{n-h-1}{b-1}\binom{n-m-(2b-1)}{r-2b-1}$$

(iii) Summing (1), (2) over $r_1+m\le h\le n-\ r+r_1$ respectively

$$\left(C^{2b}_{2n,\,2m,\,r,\,r_1}\right)_F=\binom{r_1}{b}\binom{r-r_1-1}{b-1}\sum_{h=r_1+m}^{n-r+r_1}\binom{h-m}{r_1}\binom{n-h-1}{r-r_1-1}$$

(7) $$\left(C^{2b}_{2n,\,2m,\,r,\,r_1}\right)_F=\binom{r_1}{b}\binom{r-r_1-1}{b-1}\binom{n-m}{r}\quad\text{(from II)}$$

(8) $$\left(C^{2b-1}_{2n,\,2m,\,r,\,r_1}\right)_F=\binom{r_1}{b-1}\binom{r-r_1-1}{b-1}\binom{n-m}{r}$$

(iv) Summing (1), (2) over $1\le b\le min(r_1, r-r_1)$ and $1\le b\le min(r_1+1,\ r-r_1)$ respectively

(9) $$\left(C^{(2h)}_{2n,\,2m,\,r,\,r_1}\right)^e_F=\binom{n-h-1}{r-r_1-1}\binom{h-m}{r_1}\sum_{b=1}^{min(r_1,r-r_1)}\binom{r_1}{b}\binom{r-r_1-1}{b-1}$$

(10) $$\left(C^{(2h)}_{2n,\,2m,\,r,\,r_1}\right)^o_F=\binom{n-h-1}{r-r_1-1}\binom{h-m}{r_1}\sum_{b=1}^{min(r+1,r-r_1)}\binom{r_1}{b-1}\binom{r-r_1-1}{b-1}$$

Adding (9) and (10) we get $\left(C^{(2h)}_{2n,\,2m,\,r,\,r_1}\right)_F$

(v) Summing (3) over $b\le r_1\le h-m$ we get

$$\left(C^{(2h)2b}_{2n,\,2m}\right)_F=2^{n-h-b}\binom{n-h-1}{b-1}\binom{h-m}{b}\sum_{r_1=b}^{h-m}\binom{r_1}{b}\binom{h-m}{r_1}$$

(11) $$\left(C^{(2h)2b}_{2n,\,2m}\right)_F=2^{n-m-2b}\binom{n-h-1}{b-1}\binom{h-m}{b}$$

(12) $$\left(C^{(2h)2b-1}_{2n,\,2m}\right)_F=2^{n-m-2b+1}\binom{n-h-1}{b-1}\binom{h-m}{b-1}$$

(vi) Summing (3), (4) over $1\le b\le min(r_1, n-h)$ and $1\le b\le min(r_1+1,\ n-h)$ resp.

(13) $$\left(C^{(2h)}_{2n,\,2m,\quad,\,r_1}\right)^e_F=\binom{h-m}{r_1}\sum_{b=1}^{min(r_1,\,n-h)}2^{n-h-b}\binom{r_1}{b}\binom{n-h-1}{b-1}$$

(14) $$\left(C^{(2h)}_{2n,\,2m,\quad,\,r_1}\right)^e_F=\binom{h-m}{r_1}\sum_{b=1}^{min(r_1+1,\,n-h)}2^{n-h-b}\binom{r_1}{b-1}\binom{n-h-1}{b-1}$$

Adding (13), (14) over $r_1+m\leq h\leq n-b$ respectively

$$\left(C^{2b}_{2n,\,2m,\quad,\,r_1}\right)_F=\binom{r_1}{b}\sum_{h=r_1+m}^{h-b}2^{n-h-b}\binom{h-m}{r_1}\binom{n-h-1}{b-1}$$

(15) $$\left(C^{2b}_{2n,\,2m,\,.\,.\,,\,r_1}\right)_F=\binom{r_1}{b}\sum_{i=0}^{n-b-r_1-m}2^{n-r_1-m-b-i}\binom{r_1+i}{i}\binom{n-r_1-m-i-1}{n-r_1-m-b-i}$$

(16) $$\left(C^{2b-1}_{2n,\,2m,\,.\,.\,,\,r_1}\right)_F=\binom{r_1}{b-1}\sum_{i=0}^{n-b-r_1-m}2^{n-r_1-m-b-i}\binom{r_1+i}{i}\binom{n-r_1-m-i-1}{n-r_1-m-b-i}$$

(viii) Summing (5), (6) over $1\leq b\leq min(h-m,\,n-h,\,[\frac{r}{2}])$ and $1\leq b\leq min(h-m+1,\,n-h,\,[\frac{r+1}{2}])$ resp.

(17) $$\left(C^{(2h)}_{2n,\,2m,\,r}\right)^e_F=\sum_b\binom{h-m}{b}\binom{n-h-1}{b-1}\binom{n-m-2b}{r-2b}$$

(18) $$\left(C^{(2h)}_{2n,\,2m,\,r}\right)^o_F=\sum_b\binom{h-m}{b-1}\binom{n-h-1}{b-1}\binom{n-m-(2b-1)}{r-(2b-1)}$$

(ix) Summing (5) over $b+m\leq h\leq n-b$,

$$\left(C^{2b}_{2n,\,2m,\,r}\right)_F=\binom{n-m-2b}{r-2b}\sum_{h=b+m}^{n-b}\binom{h-m}{b}\binom{n-h-1}{b-1}$$

(19) $$\left(C^{2b}_{2n,\,2m,\,r}\right)_F=\binom{n-m-2b}{r-2b}\binom{n-m}{2b}\quad\text{(from II)}$$

Similarly summing(6) over $b+m-1\leq h\leq n-b$

(20) $$\left(C^{2b-1}_{2n,\,2m,\,r}\right)_F=\binom{n-m-2b+1}{r-2b+1}\binom{n-m}{2b-1}$$

Obviously from (19), (20)

(21) $$\left(C^{b}_{2n,\,2m,\,r}\right)_F=\binom{n-m-b}{r-b}\binom{n-m}{b},\quad b\text{ may be odd or even}$$

(x) Summing (7), (8) over $1\leq b\leq min(r_1,\,r-r_1)$ and $1\leq b\leq min(r_1+1,\,r-r_1)$ respectively

(22) $$\left(C_{2n,\,2m,\,r,\,r_1}\right)^e_F=\binom{n-m}{r}\sum_{b=1}^{min(r_1,\,r-r_1)}\binom{r_1}{b}\binom{r-r_1-1}{b-1}$$

(23) $$\left(C_{2n,\,2m,\,r,\,r_1}\right)^o_F=\binom{n-m}{r}\sum_{b=1}^{min(r_1+1,\,r-r_1)}\binom{r_1}{b-1}\binom{r-r_1-1}{b-1}$$

Adding (22), (23) we get

(24) $$\left(C_{2n,\,2m,\,r,\,r_1}\right)_F=\binom{n-m}{r}\Big[\sum_{b=1}^{min(r_1,r-r_1)}\binom{r_1}{b}\binom{r-r_1-1}{b-1}+\sum_{b=1}^{min(r_1+1,r-r_1)}\binom{r_1}{b-1}\binom{r-r_1-1}{b-1}\Big]$$

(xi) Summing (11) over $b+m\leq h\leq n-b$ we get

$$\left(C^{2b}_{2n,\,2m}\right)_F=2^{n-m-2b}\sum_{h=b+m}^{n-b}\binom{n-m}{b}\binom{n-h-1}{b-1}$$

(25) $$\left(C^{2b}_{2n,\,2m}\right)_F=2^{n-m-2b}\binom{n-m}{2b}\quad\text{(from II)}$$

Also summing (12) over $b+m-1\leq h\leq n-b$

(26) $$\left(C^{2b-1}_{2n,\,2m}\right)_F=2^{n-m-2b+1}\binom{n-m}{2b-1}$$

Obviously from (25), (26) we get

(27) $$\left(C_{2n,\,2m}\right)_F=2^{n-m-b}\binom{n-m}{b},\quad b\text{ may be odd or even}$$

(xii) Summing (11), (12) over $1\leq b\leq min(h-m,n-h,[\frac{n-m}{2}])$ and $1\leq b\leq min(h-m+1,n-h,[\frac{n-m+1}{2}])$ respectively we get

(28) $$\left(C^{(2h)}_{2n,\,2m}\right)^e_F=\sum_b 2^{n-m-2b}\binom{h-m}{b}\binom{n-h-1}{b-1}$$

(29) $$\left(C^{2h}_{2n,\,2m}\right)^o_E=\sum_b 2^{n-m+2b+1}\binom{h-m}{b-1}\binom{n-h-1}{b-1}$$

(xiii) Summing (15), (16) over $1\leq b\leq min(r_1,n-r_1-m)$

(30) $$\left(C_{2n,\,2m,\,.\,,\,r_1}\right)^e_E=\sum_b\binom{r_1}{b}\sum_{i=0}^{n-b-r_1-m}2^{n-r_1-m-b-i}\binom{r_1+i}{i}\binom{n-r_1-m-i-1}{n-r_1-m-b-i}$$

(31) $$\left(C_{2n,\,2m,\,.\,,\,r_1}\right)^o_F=\sum_b\binom{r_1}{b-1}\sum_{i=0}^{n-b-r_1-m}2^{n-r_1-m-b-i}\binom{r_1+i}{i}\binom{n-r_1-m-i-1}{b-1}$$

(xiv) Summing (21) over $0\leqq b\leqq r$

$$\left(C_{2n,\,2m,\,r}\right)_F=\sum_{b=0}^{r}\binom{n-m-b}{r-b}\binom{n-m}{b}$$

$$=\binom{n-m}{r}\sum_{b=0}^{r}\binom{r}{b}$$

(32) $$\left(C_{2n,\,2m,\,r}\right)_F=2^r\binom{n-m}{r}$$

(xv) Summing (32) over $0\leq r\leq n-m$

(33) $$\left(C_{2n,\,2m}\right)_F=3^{n-m}$$

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