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ON FLUCTUATIONS IN COIN TOSSING AND RANDOM WALK

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In one dimensional random walk, Chung and Feller [1] considered the problem of coin tossing game by taking a sequence of independent random variables X_j, $j=1,2,\cdots$ each taking value ± 1 with probabilities $\frac{1}{2}$ and studied the behaviour of partial sums $S_n = \sum_{j=1}^{n} X_j$ by making the convention: 'S_n' is positive if $S_n>0$ or if $S_n=0$ but $S_{n-1}>0$ other wise 'S_n' is negative. This problem was further studied by Feller [3], Csaki and Vincze [2]; Kanwar Sen [5] and Jain [4].

We now study the problem by further defining an event 'E' for sequence of partial sums $\{S_n\}$ that:
If $S_r=0$ for $r=2\alpha_1, 2\alpha_2, \cdots 2\alpha_i=2n$; $i=1,2,\cdots n$, then the j^{th} segment included between the two consecutive zeros i.e. between the $(j-1)^{th}$ and j^{th} zero, satisfy the condition

$$
'E' \begin{cases} 0=S_{2\alpha_{j-1}}<S_{2\alpha_{j-1}+1}<S_{2\alpha_{j-1}+2}<\cdots<S_{\alpha_j-\alpha_{j-1}}> \\ >S_{\alpha_j-\alpha_{j-1}+1}>S_{\alpha_j-\alpha_{j-1}+2}>\cdots>S_{2\alpha_j}=0 \\ \text{or} \\ 0=S_{2\alpha_{j-1}}>S_{2\alpha_{j-1}+1}>S_{2\alpha_{j-1}+2}>>S_{\alpha_j-\alpha_{j-1}}< \\ <S_{\alpha_j-\alpha_{j-1}+1}<S_{\alpha_j-\alpha_{j-1}+2}<\cdots<S_{2\alpha_j}=0 \end{cases}
$$

for $j=1,2,\cdots i$ and $i=1,2,\cdots n$.

Alternatively the event E can be described as "the sequence $\{S_0, S_1, \cdots S_n\}$ does not satisfy the condition: $S_{j-1}>S_j<S_{j+1}$ when $S_j>0$ and $S_{j-1}<S_j>S_{j+1}$ when $S_j<0$ for all j.

For a simple random walk starting from the origin and returning to $(2n, 0)$, the event E can be interpreted as: A particle goes up (comes down) as many steps as it likes but once it comes down (goes up) it has to come down (go up) continuously some number of steps it went up (came down) until it reaches the starting level. In next step it may either go up or come down and follows the above pattern i.e. if it goes up (comes down) it moves upwards (downwards) but once it starts coming down (going up) it will continuously come down (go up) until it again reaches at the starting level. In other words it may also be defined as "the particle approaching the origin does not revert its direction until it reaches the origin".

Also in gambling terminology, it can be interpreted as: A gambler starts winning (losing) and goes on winnig (losing) but once he starts losing (winning) he goes on

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losing (winning) continuously until his net gain is zero. In the next trial he may either win or lose and then will follow the above pattern i.e. if he wins (loses) he goes on winning (losing) but once he starts losing (winning) he will continuously lose (win) until his net gain is again zero. Thus the event 'E' may be described as "if the gambler's gain (loss) is approaching (or decreasing to) zero, it can't increase in any trial unless it reduces to zero".

In this paper we obtain the probability that a particle starting from the origin and returning to the point $(2n, 0)$ at the $2n^{th}$ step satisfying the event 'E' and having

(1) r returns to the x-axis

(2) $2b$ crossings with the x-axis

(3) r returns, $2b$ crossings with the x-axis

(4) r returns, $2b$ crossings and $2h$ steps above x-axis.

Path Representation:

Let X_j be a $r.v.$ associated with the i^{th} step of the simple random walk, taking two values ± 1 according to as the particle has a positive and negative step respectively. Writing $S_0=0$, $S_j=X_1+X_2+\cdots+X_j (j>0)$ the So, $S, \cdots S_n$ satisfy the condition $S_j-S_{j-1}=X_j=\pm 1$; $j=1, 2, \cdots n$. Using the geometrical terminology, the points (j, S_j) when plotted on a x-y plane and joined successively by st. line segments, we get a path whose vertices have absicissa $0, 1, \cdots n$ and ordinates $S_0, S_1, \cdots S_n$ respectively. Such a path may be taken as representing the simple random walk.

Notations:

A-points	: a point (j, S_j) with $S_j=0$ i.e. a return to the x-axis.
$A^+(A^-)$	: an A-point s.t. $S_{j-1}=+1$ $(S_{j-1}=-1)$. It is a positive (negative) return point.
V(wave)	: a segment of a path included between two consecutive A-points. The segment from origin to the first return point is also regarded as a wave.
$V^+(V^-)$	: a wave (V) with $S_j>0$ $(S_j<0)$ at the intervening position.
B(crossing or intersection with x-axis)	: a point (i, S_j) of the path with $S_j=0$ and $S_{j-1}\cdot S_{j+1}=-1$
C(section)	: a segment of a path included betweentwo consecutive B-points. The segments from origin to the first B point and that from the last B-point to the end point are also regarded as sections.
$C^+(C^-)$	: a section C with $S_j\geq 0$ $(S_j\leq 0)$ in between.
C_n	: a path $S_0, S_1, \cdots S_n$ with $S_0=S_n=0$ with n even
C_n^b	: a C_n with b B-points.
$C_n^b(+)(C_n^b(-))$	: a C_n^b with $S_1=+1$ $(S_1=-1)$
$C_{n,r}$	: a C_n with r A-points.
$C_{n,\cdot,r_1}$	: a C_n with r_1 A^+-points.
$C^b_{n,r}$	: a C_n^b with r A-points.
$C^b_{n,\cdot r}$	: a C_n^b with r_1 A^+-points.
C_{n,r,r_1}	: a C_n with r A-points, r_1 A^+-points.

C^b_{n,r,r_1} : a C_{n,r_1r} with b B-points.
$C_n^{(h)b}$: a C_n^b with h steps above x-axis.
$C^{(h)}_{n,r}$: a $C_{n,r}$ with h steps above x-axis.
$C^{(h)}_{n,r,r_1}$: a C_{n,r,r_1} with h steps above x-axis.
$C^{(h)b}_{n,r,r_1}$: a C^b_{n,r,r_1} with h steps above x-axis.
$(\ldots)_E$: No. of possible paths of the type$\cdots$ which all satisfy the event E.

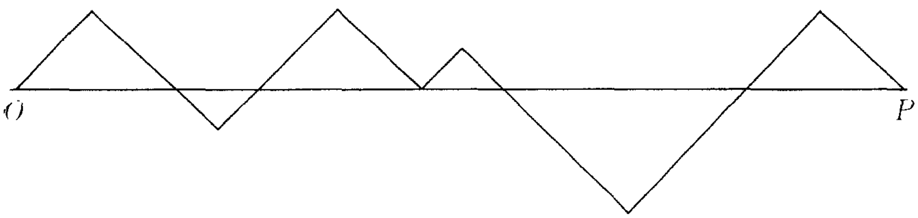
Theorem:

(1) $$(C^{(2h)2b}_{2h,r,r_1}(+))_E=\binom{h-1}{r_1-1}\binom{r_1-1}{b}\binom{n-h-1}{r-r_1-1}\binom{r-r_1-1}{b-1}$$

(2) $$(C^{(2h)2b}_{2n,r,r_1}(-))_E=\binom{h-1}{r_1-1}\binom{r_1-1}{b-1}\binom{n-h-1}{r-r_1-1}\binom{r-r_1-1}{b}$$

(3) $$(C^{(2h)2b-1}_{2n,r,r_1})_E=2\binom{h-1}{r_1-1}\binom{r_1-1}{b-1}\binom{n-h-1}{r-r_1-1}\binom{r-r_1-1}{b-1}$$

Proof.



Let OP be a $C^{(2h)2b}_{2n,r,r_1}$ path with $S_1=+1$ (Fig$\cdots$) as envisaged in (1). Since $S_1>0$ and $S_{2n-1}>0$. This path would consist of r_1 V^+ of total length $2h$ steps giving rise to $(b+1)$ C^+ and $(r-r_1)V^-$ of length $(2n-2h)$ steps constituting bC^-. r_1V^+ of length $2h$ steps can be constructed in $\binom{h-1}{r_1-1}$ ways since each V^+ has to be of length at least 2 steps and $\binom{r_1-1}{b}$ is the number of ways of constructing $(b+1)C^+$ out of r_1V^+. This is akin to distributing g similar balls into C district cells which is possible in $\binom{g-1}{C-1}$ ways. Therefore $(b+1)C^+$ can be constructed from r_1V^+ of length $2h$ steps in $\binom{h-1}{r_1-1}\binom{r_1-1}{b}$ ways. Similarly $\binom{h-h-1}{r-r_1-1}\binom{r-r_1-1}{b-1}$ is the number of ways of forming bC^- out of $(r-r_1)V^-$

of length $(2n-2h)$ steps. Thus joining the waves in order we get the required number of paths

$$\left(C_{2n,r,r_1}^{(2h)2b}(+)\right)_E=\binom{h-1}{r_1-1}\binom{r_1-1}{b}\binom{n-h-1}{r-r_1-1}\binom{r-r_1-1}{b-1}$$

Now we consider the path of the type $\left(C_{2n,r,r_1}^{(2h)2b}\right)$ satisfying event E and having $S_1=-1$. Such a path consists of r_1V^+ of length $2h$ steps constituting bC^+ and $(r-r_1)V^-$ of length $(2n-2h)$ steps forming $(b+1)C^-$. Therefore as in (1), we easily get (2).

Similarly proceeding for $\left(C_{2n,r,r_1,}^{(2h)2b-1}\right)$ it is obvious that number of paths remain the same whether $S_1=+1$ or $S_1=1$ which accounts for the factor 2 in (3). This path has $bC^+$and bC^- consisting of r_1V^+ of length $2h$ steps and $(r-r_1)V^-$ of length $(2n-2h)$ steps respectively. Thus arguing as for (1) we easily get (3).

Deductions:

(i) Summing (1) ever $b+r_1\leq r\leq n-h+r_1$ we get

$$(C_{2n,\cdot,r_1}^{(2h)2b}(+))_E=\sum_{r=b+r_1}^{n-h+r_1}\binom{h-1}{r_1-1}\binom{r_1-1}{b}\binom{n-h-1}{r-r_1-1}\binom{r-r_1-1}{b-1}$$

$$=\binom{h-1}{b}\binom{n-h-1}{b-1}\binom{h-b-1}{n-r_1}\sum_{r=b+r_1}^{n-h+r_1}\binom{n-h-b}{r-r_1-b}$$

(4) $$\left(C_{2n,\cdot,r_1}^{(2h)}(+)\right)_E=2^{n-h-b}\binom{h-1}{r_1-1}\binom{r_1-1}{b}\binom{n-h-1}{b-1}$$

(ii) Summing (1) over $b+1\leq r_1\leq r-b$

$$\left(C_{2n,r}^{(2h)2b}(+)\right)_E=\sum_{r_1=b+1}^{r-b}\binom{h-1}{r_1-1}\binom{r_1-1}{b}\binom{n-h-1}{r-r_1-1}\binom{r-r_1-1}{b-1}$$

$$=\binom{h-1}{b}\binom{n-h-1}{b-1}\sum_{r=b+1}^{-lr}\binom{h-b-1}{h-r_1}\binom{n-h-b}{r-r_1-b}$$

$$=\binom{h-1}{b}\binom{n-h-1}{b-1}\binom{n-2b-1}{r-2b-1}$$

(5) $$\left(C_{2n,r}^{(2h)2b}(+)\right)_E=\binom{h-1}{b}\binom{n-h-1}{b-1}\binom{n-2b-1}{n-r}$$

(iii) Again summing (1) over $r_1\leq h\leq n-r+r_1$

$$\left(C_{2n,r,r_1}^{2b}(+)\right)_E=\binom{r_1-1}{b}\binom{r-r_1-1}{b-1}\sum_{h=r_1}^{n-r+r_1}\binom{h-1}{r_1-1}\binom{n-h-1}{r-r_1-1}$$

(6) $$\left(C_{2n,r,r_1}^{2b}(+)\right)_E=\binom{r-1}{b}\binom{r-r_1-1}{b-1}\binom{n-1}{r-1}$$

(iv) Summing (1) over possible values of b,

(7) $$\left(C^{(2h)}_{2n,\,r,\,r_1}(+)\right)_{E^e}=\binom{h-1}{r_1-1}\binom{n-h-1}{r-r_1-1}\sum_{b=1}^{\min(r_1-1,\,r-r_1)}\binom{r_1-1}{b}\binom{r-r_1-1}{b-1}$$

where superscript 'e' stands for the even numer of crossings.

(v) Summing (4) over $b+1\leq r_1\leq h$ we get

$$\left(C^{(2h)\,2b}_{2n}(+)\right)_E=\binom{n-h-1}{b-1}\sum_{r_1=b+1}^{h}2^{n-h-b}\binom{h-1}{r_1-1}\binom{r_1-1}{b}$$

$$=2^{n-h-b}\binom{n-h-1}{b-1}\binom{h-1}{b}\sum_{r_1=b+1}^{h}\binom{h-b-1}{r_1-b-1}$$

(8) $$\left(C^{(2h)\,2b}_{2n}(+)\right)_E=2^{n-2b-1}\binom{h-1}{b}\binom{n-h-1}{b-1}$$

(vi) Summing (4) again over $1\leq b\leq \min\,(r_1-1,\ n-h)$

(9) $$\left(C^{(2h)}_{2n,\,\cdot\,,\,r_1}(+)\right)_E=\binom{h-1}{r_1-1}\sum_{b=1}^{\min(r_1-1,\,n-h)}2^{n-h-b}\binom{r_1-1}{b}\binom{n-h-1}{b-1}$$

(vii) Summing (4) over $r_1\leq h\leq n-b$

$$\left(C^{2b}_{2n,\,\cdot\,,\,r_1}(+)\right)_E=\binom{r_1-1}{b}\sum_{h=r_1}^{n-b}2^{n-h-b}\binom{h-1}{r_1-1}\binom{n-h-1}{b-1}$$

$$=\binom{r_1-1}{b}\sum_{i=0}^{n-b-r_1}2^{n-r_1-b-i}\binom{r_1+i-1}{i}\binom{n-r_1-i-1}{b-1}$$

(10) $$\left(C^{2b}_{2n,\,\cdot\,,\,r_1}(+)\right)_E=\binom{r_1-1}{b}\sum_{i=0}^{n-b-r_1}2^{n-r_1-b-i}\binom{r_1+i-1}{i}\binom{n-r_1-i-1}{b-1}$$

(viii) Summing (7) over $1\leq b\leq\min(h-1,\ n-h,\ [\frac{r-1}{2}])$

(11) $$\left(C^{(2h)}_{2n,\,r}(+)\right)_E=\sum_b\binom{h-1}{b}\binom{n-h-1}{b-1}\binom{n-2b-1}{n-r}$$

(ix) Summing (5) over $b+1\leq h\leq n-b$

$$\left(C^{2b}_{2n,\,r}(+)\right)_E=\binom{n-2b-1}{n-r}\sum_{h=b+1}^{n-b}\binom{h-1}{b}\binom{n-h-1}{b-1}$$

$$=\binom{n-2b-1}{n-r}\binom{n-1}{2b}$$

(12) $$\left(C^{2b}_{2n,\,r}(+)\right)_E=\binom{n-2b-1}{n-r}\binom{n-1}{2b}=\binom{n-1}{r-1}\binom{r-1}{2b}$$

Similarly summing (2) and (3) over possible values of r_1 and h respectively we get

(13) $$\left(C^{2b}_{2h,\,r}(-)\right)_E=\binom{n-1}{r-1}\binom{r-1}{2b}$$

And (14) $$\left(C_{2n,r}^{2b-1}\right)_E=2\binom{n-1}{r-1}\binom{r-1}{2b-1}$$

Obviously from (12), (13) and (14) we get

(15) $$\left(C_{2n,r}^{b}\right)_E=2\binom{n-1}{r-1}\binom{r-1}{b}, \text{ } b \text{ may be odd or even}$$

(x) Summing (6) over $1\le b\le \min\ (r_1-1,\ r-r_1)$

(16) $$\left(C_{2n,r,r_1}(+)\right)_E=\binom{n-1}{r-1}\sum_b\binom{r_1-1}{b}\binom{r-r_1-1}{b-1}$$

(xi) Summing (8) over $b+1\le h\le n-b$ we get

$$C_{2n}^{2b}(+)=2^{n-2b-1}\sum_{h=b+1}^{n-b}\binom{h-1}{b}\binom{n-h-1}{b-1}$$

(17) $$\left(C_{2n}^{2b}(+)\right)_E=2^{n-2b-1}\binom{n-1}{2b}$$

(xii) Summing (10) over possible values of b

(18) $$\left(C2n,.,r_1(+)\right)_E=\sum_{b=1}^{\min(r_1-1,n-r_1)}\binom{r_1-1}{b}2^{n-r_1-b-i}\binom{r_1+i-1}{r_1-1}\binom{n-r_1-i-1}{b-1}$$

(xiii) Summing (8) over $1\le b\le \min(h-1, n-h,\ [\frac{n-1}{2}])$

(19) $$\left(C_{2n}^{(2h)}(+)\right)_E=\sum_b 2^{n-2b-1}\binom{h-1}{b}\binom{n-h-1}{b-1}$$

(xiv) Summing (15) over $0\le b\le r-1$

$$\left(C_{2n,r}\right)_E=2\binom{n-1}{r-1}\sum_{b=0}^{r-1}\binom{r-1}{b}$$

$$=2\binom{n-1}{r-1}2^{r-1}$$

(20) $$\left(C2n,r\right)_E=2^r\binom{n-1}{r-1}$$

(xv) Summing (20) over $1\le r\le n$

$$\left(C_{2n}\right)_E=\sum_{r=1}^{n}2^r\binom{n-1}{r-1}$$

$$=2\ 3^{n-1}$$

Similar results can be obtained when $S_1=-1$ and also for odd number of crossings since we considered only even number of crossings in above deductions.

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