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INDUCED TRANSFORMATIONS ON EXTERIOR PRODUCT SPACES

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1. Introduction

Let E be an n-dimensional vector space over a field F and let $L(E, E)=L(E)$ be the set of all linear transformations of E. A linear transformation h in $L(E)$ induces transformations of $\Lambda^p E$, the space of p-vectors on E. A. P. Stone ([6]) obtained an identity which gives a relation among the induced transformations. We generalize this identity. h in $L(E)$ induces the endomorphism h^Λ of ΛE, the exterior algebra over E. We obtain a formula for the rank of an endomorphism $(gh)^\wedge$, where g, h in $L(E)$.

2. The induced transformations $h^{(q)}$.

Let $\Lambda E=\Lambda^0 E\oplus\Lambda^1 E\oplus\cdots\oplus\Lambda^n E=\bigoplus_{p=0}^{n}\Lambda^p E$ be the *exterior algebra* over a vector space E, where $\Lambda^0 E=F$, $\Lambda^1 E=E$ and $\Lambda^p E$ is the *p-th exterior power* of E $(1\leq p\leq n)$ ([1]). If $\{e_1, e_2, \cdots, e_n\}$ is a basis of E, then $\{e_{i_1}\Lambda\cdots\Lambda e_{i_p}\,|\,1\leq i_1<\cdots<i_p\leq n\}$ is a basis of $\Lambda^p E$ and $\Lambda^p E$ is a vector space of dimension $\binom{n}{p}$ over F. Let $S(i)$ be the set of all permutations of the set $\{i_1, \cdots i_p\}$.

A linear transformation h in $L(E)$ induces well-defined linear transformations $h^{(q)}$: $\Lambda^p E\to\Lambda^p E$, $0\leq q\leq p\leq n$, which are obtained by

$$h^{(q)}(e_{i_1}\Lambda\cdots\Lambda e_{i_p})=\frac{1}{(p-q)!q!}\sum_{\sigma\in S(i)}|\sigma|\,(he_{\sigma(i_1)}\Lambda\cdots\Lambda he_{\sigma(i_q)}\Lambda e_{\sigma(i_{q+1})}\Lambda\cdots\Lambda e_{\sigma(i_p)}),$$

where $|\sigma|$ denotes the signature of the permutation σ and the transformation $h^{(0)}$ is taken to be the identity on $\Lambda^p E$. If $p<q$ then it is understood that $h^{(q)}x=0$ for any p-vector x in $\Lambda^p E$.

For $h\in L(E)$ we put $h\cdots h$(q-times) by h^q. Let $\{j_1, \cdots, j_q\}$ be a set of positive integers, and we define a linear transformation $[h^{j_1}, \cdots, h^{j_q}]$: $\Lambda^p E\to\Lambda^p E$ as follows. For each basis element $e_{i_1}\Lambda\cdots\Lambda e_{i_p}(1\leq i_1<\cdots<i_p\leq n)$ of $\Lambda^p E$, define

$$[h^{j_1}, \cdots, h^{j_q}](e_{i_1}\Lambda\cdots\Lambda e_{i_p})$$

$$=\frac{1}{(p-q)!q!}\sum_{\sigma\in S(i)}|\sigma|\,(h^{j_1}e_{\sigma(i_1)}\Lambda\cdots\Lambda h^{j_q}e_{\sigma(i_q)}\Lambda e_{\sigma(i_{q+1})}\Lambda\cdots\Lambda e_{\sigma(i_p)})$$

for $0\leq q\leq p\leq n$, and $[h^{j_1}, \cdots, h^{j_q}]=0$ for $p<q$.

We also put $[\prod_{r_1} h_1, \cdots, \prod_{r_k} h_k]=[\overbrace{h_1, \cdots, h_1}^{r_1\text{-times}}, \cdots, \overbrace{h_k, \cdots, h_k}^{r_k\text{-times}}]$, where $h_1, \cdots$, and h_k are in $L(E)$, and $r_1, \cdots, r_k$ are positive integers.

For example,

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$$h^{(q)}=[\overbrace{h,\cdots,h}^{q\text{-times}}]=[\prod_q h],\quad [h^2]^{(4)}=[h^2,h^2,h^2,h^2]=[\prod_4 h^2] \text{ and}$$

$$[h^3,h^3,h^2,h^2,h^2,h,h,h,h,h]=[\prod_2 h^3,\prod_3 h^2,\prod_5 h],\quad \text{where } h\in L(E).$$

LEMMA 1. *Let $h\in L(E)$. (1) $[h^i,h^j]=[h^j,h^i]$ for positive integers i and j. (2) For a set $\{j_1,\cdots,j_q\}$ of positive integers and a permutation σ of $\{1,2,\cdots,q\}$*

$$[h^{j_1},\cdots,h^{j_p}]=[h^{j_{\sigma(1)}},\cdots,h^{j_{\sigma(q)}}].$$

Proof. If $x\in\Lambda^0E\oplus\Lambda^1E$, then clearly $[h^i,\ h^j]x=0=[h^j,h^i]x$. Let $e_{i_1}\Lambda\cdots\Lambda e_{i_p}$ be an element in a basis for Λ^pE. Then, by our definition, it is easy to see that $[h^i,\ h^j]\ (e_{i_1}\Lambda\cdots\Lambda e_{i_p})=[h^j,\ h^i]\ (e_{i_1}\Lambda\cdots\Lambda e_{i_p})$ for $p\geq 2$. Similarly, we can see that $[h^i,h^j]x=[h^j,h^i]x$ for any member x in a basis for Λ^pE. Since $p\geq 2$ is arbitrary, (1) is established. The proof of (2) is similar to the proof of (1).

THEOREM 2. *Let $x\in\Lambda^pE$ and let $q=mk+r$, where q, $m+1$, k and r are positive integers. Let*

$$L=(h^q)^{(k)}-(h^{q-k})^{(k)}(h^k)^{(k)}+\cdots+(-1)^i(h^{q-ik})^{(k)}(h^k)^{(ik)}$$
$$+\cdots+(-1)^{m-1}(h^{k+r})^{(k)}(h^k)^{((m-1)k)}+(-1)^m(h^r)^{(k)}(h^k)^{(mk)}$$

and let

$$L(i,t)=(-1)^i\binom{k}{t}\binom{ki+t}{k}[\prod_{k-t}h^{k(m-i+1)+r},\prod_t h^{k(m-i)+r},\prod_{k(i-1)+t}h^k],$$

where $\binom{i}{j}$ denotes the binomial coefficient for two positive integers i and j. Then

(i) $Lx=0$ if $k>p$,

(ii) $Lx=(\sum_{t=1}^{k_0}\sum_{i=1}^{u}L(i,t)+\sum_{t=k_0+1}^{k-1}\sum_{i=1}^{u-1}L(i,t))x$ if $k\leq p=uk+k_0<(m+1)k$,

where $0\leq k_0<k$,

(iii) $Lx=(\sum_{t=1}^{k-1}\sum_{i=1}^{m}L(i,t))x+L(m,k)x$ if $q\leq(m+1)k\leq p$.

Proof. It is sufficient to prove the identity for all elements of a basis of Λ^pE, because any element in Λ^pE can be expressed as a linear combination of elements in a basis of Λ^pE. We first prove (iii): Let $\{e_1,e_2,\cdots,e_n\}$ be a basis for E. Then for any set $\{j_1,\cdots,j_p\}$ of positive integers with $1\leq j_1<j_2<\cdots<j_p\leq n$, $e_{j_1}\Lambda\cdots\Lambda e_{j_p}$ is a member of a basis of Λ^pE. We just prove the identity (iii) for $e_1\Lambda\cdots\Lambda e_p$ only, because the proof of the identity for any other basis element is technically the same as the proof for $e_1\Lambda\cdots\Lambda e_p$. Consider

$$((h^{q-ik})^{(k)}(h^k)^{(ik)}-(h^{q-(i+1)k})^{(k)}(h^k)^{((i+1)k)}(e_1\Lambda\cdots\Lambda e_p).$$

An expansion of $(h^{q-ik})^{(k)}(h^k)^{(ik)}(e_1\Lambda\cdots\Lambda e_p)$ leads to $\binom{p}{ik}\binom{p}{k}$ terms of $(k+1)$ types. There-are $\binom{p}{ik}\binom{ik}{k}$ distinct terms of the form

$$h^{q-(i-1)k}e_{j_1}\Lambda\cdots\Lambda h^{q-(i-1)k}e_{j_k}\Lambda h^k e_{j_{k+1}}\Lambda\cdots\Lambda h^k e_{j_{ik}}\Lambda e_{j_{ik+1}}V\cdots\Lambda e_{j_p}.$$

We rewrite the above in the following:

$$|[\prod_k h^{q-(i-1)k}, \prod_{(i-1)k} h^k]| = \binom{p}{ik}\binom{ik}{k}.$$

Similarly, we have that

$$|[\prod_{k-1} h^{q-(i-1)k},\ h^{q-ik},\ \prod_{(i-1)k+1} h^k]| = \binom{p}{ik}\binom{ik}{k-1}\binom{p-ik}{1}, \cdots\cdots,$$

$$|[\prod_{k-t} h^{q-(i-1)k},\ \prod_t h^{q-ik},\ \prod_{(i-1)k+t} h^k]| = \binom{p}{ik}\binom{ik}{k-t}\binom{p-ik}{t}, \cdots\cdots,$$

$$|[[h^{q-(i-1)k},\ \prod_{k-1} h^{q-ik},\ \prod_{ik-1} h^k]| = \binom{p}{ik}\binom{ik}{1}\binom{p-ik}{k-1},$$

$$|[\prod_k h^{q-ik},\ \prod_{ik} h^k]| = \binom{p}{ik}\binom{p-ik}{k}.$$

Similarly, the expansion of $(h^{q-(i+1)k})^{(k)}(h^k)^{((i+1)k)}$ $(e_1\Lambda\cdots\Lambda e_p)$ leads to $\binom{p}{(i+1)k}\binom{p}{k}$ terms of $(k+1)$ types. We also have the following:

$$|[\prod_k h^{q-ik},\ \prod_{ik} h^k]| = \binom{p}{(i+1)k}\binom{(i+1)k}{k},$$

$$|[\prod_{k-1} h^{q-ik}, h^{q-(i+1)k},\ \prod_{ik+1} h^k]| = \binom{p}{(i+1)k}\binom{(i+1)k}{k-1}\binom{p-(i+1)k}{1}, \cdots\cdots,$$

$$|[\prod_{k-t} h^{q-ik},\ \prod_t h^{q-(i+1)k},\ \prod_{ik+t} h^k]| = \binom{p}{(i+1)k}\binom{(i+1)k}{k-t}\binom{p-(i+1)k}{t}, \cdots\cdots,$$

$$|[\prod_k h^{q-(i+1)k},\ \prod_{(i+1)k} h^k]| = \binom{p}{(i+1)k}\binom{(p-(i+1)k}{k}.$$

It is easy to check that each of $(h^{q-ik})^{(k)}(h^k)^{(ik)}(e_1\Lambda\cdots\Lambda e_p)$ and $(h^{q-(i+1)k})^{(k)}(h^k)^{((i+1)k)}$ $(e_1\Lambda\cdots\Lambda e_p)$ has the term of the form

$$h^{q-ik}e_{j_1}\Lambda\cdots\Lambda h^{q-ik}e_{j_k}\Lambda h^k e_{j_{k+1}}\Lambda\cdots\Lambda h^k e_{j_{(i+1)k}}\Lambda e_{j_{(i+1)k+1}}\Lambda\cdots\Lambda e_{j_p}.$$

Hence, noting that $\binom{p}{ik}\binom{p-ik}{k} = \binom{p}{(i+1)k}\binom{(i+1)k}{k}$, the terms mentioned in above will be cancelled in the expansion. Note that $(h^q)^{(k)}(e_1\Lambda\cdots\Lambda e_p)$ leads to $\binom{p}{k}$ terms of a type of the form $h^q e_{j_1}\Lambda\cdots\Lambda h^q e_{j_k}\Lambda e_{j_{k+1}}\Lambda\cdots\Lambda e_{j_p}$ which will be cancelled in the expansion of Lx in the theorem. For the coefficient $(-1)^i\binom{k}{t}\binom{ki+t}{k}$ of the expression

$$L(i,t)x=(-1)^i\binom{k}{t}\binom{ki+t}{k}[\prod_{k-t}h^{k(m-i+1)+r},\ \prod_{t}h^{k(m-i)+r},\ \prod_{ik-k+t}h^k]x,$$

note that there are $ki+t$ elements in the square bracket of

$$|[\prod_{k-t}h^{q-(i-1)k},\ \prod_{t}h^{q-ik},\ \prod_{(i-1)k+t}h^k]|=\binom{p}{ik}\binom{ik}{k-t}\binom{p-ik}{t}$$

and $\binom{p}{ik}\binom{ik}{k-t}\binom{p-ik}{t}=\binom{p}{ki+t}\binom{ki+t}{k}\binom{k}{t}$. Thus we get the coefficient $\binom{k}{t}\binom{ki+t}{k}$ in the expression $L(i,t)(e_1\Lambda\cdots\Lambda e_p)$. Hence the identity (iii) of the theorem is established.

Proof of (i) : If $k>p$ then $Lx=0$ by our definition.

Proof of (ii) : If $k\le p=uk+k_0<(m+1)k$ and $0\le k_0<k$, the identity in (ii) follows from the identity for $q\le(m+1)k\le p$ by considering a part of the right hand side of the identity in (iii).

3. The rank of $(gh)^\Lambda$.

Each linear transformation h in $L(E)$ induces the linear transformation $h^\Lambda=h^{(0)}\oplus h^{(1)}\oplus\cdots\oplus h^{(n)}$ of ΛE, where $h^{(0)}$ is taken to be the identity map on F. Then $(hg)^\Lambda=h^\Lambda g^\Lambda$ for h and g in $L(E)$ ([1]). Let $L(\Lambda E,\Lambda E)=L(\Lambda E)$ be the set of all linear transformations of ΛE. Then there exists the map $\Lambda: L(E)\to L(\Lambda E)$ defined by $\Lambda(h)=h^\Lambda$ for each $h\in L(E)$. In this case Λ is a monomorphism and $Im\Lambda\subsetneqq L(\Lambda E)$.

PROPOSITION 3. *For h, g in $L(E)$*

$$\text{rank }((gh)^\Lambda)=2^{\text{rank}(h)-\dim(Im\,h\cap Ker\,g)}$$

Proof: Since $\text{rank}(h^\Lambda)=2^{\text{rank}(h)}$ for h in $L(E)$ ([4]), our proposition is clear ([3], [4]).

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