J. Korean Math. Soc.
Vol. 13, No. 1, 1976

ON CONFORMAL Q-KILLING TENSORS IN A QUATERNION KAEHLERIAN MANIFOLD

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§ 0. Introduction.

In n-dimensional Riemannian manifold M, a vector field v_a is called a Killing vector if it satisfies the Killing equation:

$$\nabla_a v_b + \nabla_b v_a = 0,$$

where ∇ denotes the operator of the covariant differentiation with respect to the Riemannian connection.

A Killing tensor v_{cd} is, by definition, a skew symmetric tensor satisfying Killing-Yano's equation:

$$\nabla_b v_{cd} + \nabla_c v_{bd} = 0.$$

S. Tachibana defined a conformal Killing tensor u_{cd} in M by the condition

$$\nabla_b u_{cd} + \nabla_c u_{bd} = 2\rho_d g_{bc} - \rho_c g_{bd} - \rho_b g_{cd},$$

where ρ_c is a vector field and g_{bc} the Riemannian metric, and obtained the following theorems [4].

THEOREM A. *If there exists (locally) a conformal Killing tensor which takes any preassigned (skew symmetric) value at any point of an n (>3)-dimensional Riemannian space, then the space is conformally flat.*

THEOREM B. *In a Riemannian space M (dim. $M>2$) of constant curvature with $k=R/n(n-1)\neq 0$, a conformal Killing tensor u_{cd} is uniquely decomposed in the form:*

$$u_{cd} = p_{cd} + q_{cd}, \tag{0.1}$$

where p_{cd} is a Killing tensor and q_{cd} is a closed conformal Killing tensor. In this case q_{cd} is of the form

$$q_{cd} = (-1/k)\nabla_c \rho_d,$$

where ρ_d is the associated vector of u_{cd}.

Conversely if p_{cd} is a Killing tensor and ρ_d is a Killing vector, then u_{cd} given by (0.1) is a conformal Killing tensor.

In the present paper we shall define a conformal Q-Killing tensor in a quaternion Kaehlerian manifold as a generalization of the conformal Killing tensor in a Riemannian

Received by the editors Jan. 20, 1976.

manifold.

The purpose of the present paper is to obtain the results corresponding to the theorem A and to the theorem B in a quaternion Kaehlerian manifold.

§ 1. Preliminaries.

Let M be an almost quaternion manifold, that is, a $4n$-dimensional differentiable manifold which admits a set of three tensor fields F, G, H of type (1.1) satisfying

$$F^2=-I, \quad G^2=-I, \quad H^2=-I, \tag{1.1}$$
$$F=GH=-HG, \quad G=HF=-FH, \quad H=FG=-GF,$$

I denoting the identity tensor.

In a previous paper [5], we proved that there exists a Hermitian metric g for the almost quaternion structure F, G, H, that is, a Riemannian metric satisfying

$$g(FX, \ FY)=g(X, Y),$$
$$g(GX, \ GY)=g(X, Y), \tag{1.2}$$
$$g(HX, \ HY)=g(X, Y)$$

for arbitrary vector fields X and Y of M. In this case M is called an almost quaternion metric manifold.

If an almost quaternion metric manifold M satisfies the condition

$$\nabla_X F= \ r(X)G-q(X)H,$$
$$\nabla_X G= -r(X)F+p(X)H, \tag{1.3}$$
$$\nabla_X H= \ q(X)F-p(X))G.$$

where ∇ is the operater of covariant differentiation with respect to g, p, q, r certain 1-forms and X an arbitrary vector field of M, then M is called a quaternion Kaehlerian manifold [3].

Let $\{U; x^h\}$ be a system of coordinate neighborhoods of M, where, here and in the sequel, the indices $h, i, j, k, \cdots$ run over the range $\{1, 2, \cdots, 4n\}$.

Since a quaternion Kaehlerian manifold is an Einstein space [3], the Ricci tensor has components of the form:

$$K_{ji}=4k(n+2)g_{ji}, \tag{1.4}$$

where $4n=\dim. M$. In this case, using the Ricci formula, we obtain from (1, 3)

$$K_{kjt}{}^h F_i{}^t - K_{kji}{}^t F_t{}^h = 4k(G_{kj}H_i{}^h - H_{kj}G_i{}^h),$$
$$K_{kjt}{}^h G_i{}^t - K_{kji}{}^t G_t{}^h = 4k(H_{kj}F_i{}^h - F_{kj}H_i{}^h), \tag{1.5}$$
$$K_{kjt}{}^h H_i{}^t - K_{kji}{}^t H_t{}^h = 4k(F_{kj}G_i{}^h - G_{kj}F_i{}^h),$$

where $K_{kji}{}^{t}$ is the curvature tensor of M.

In a quaternion Kaehlerian manifold M, the following formulas are known [3].

$$(1.6)\qquad \begin{aligned} &K_{kjts}F^{ts}=-8knF_{kj},\\ &K_{kjts}G^{ts}=-8knG_{kj},\quad K_{kjts}H^{ts}=-8knH_{kj},\end{aligned}$$

$$(1.7)\qquad \begin{aligned} &K_{ktsj}F^{ts}=4knF_{kj},\\ &K_{ktsj}G^{ts}=4knG_{kj},\quad K_{ktsj}H^{ts}=4knH_{kj}.\end{aligned}$$

Moreover, S, Ishihara proved the following [3]

THEOREM C. *A quaternion Kaehlerian manifold M (dim. $M\geqq 8$) is of constant Q-sectional curvature $4k$ if and only if its curvature tensor has components of the form*

$$(1.8)\qquad K_{kjih}=k(g_{kh}g_{ji}-g_{ki}g_{jh}+Q_{khji}-Q_{kijh}-2Q_{kjih}),$$

where the tensor Q_{khji} is defined by

$$(1.9)\qquad Q_{khji}=F_{kh}F_{ji}+G_{kh}G_{ji}+H_{kh}H_{ji}.$$

§ 2. Conformal Q-Killing tensor.

In a previous paper [2], we defined a Q-Killing tensor in a $4n$-dimensional quaternion Kaehlerian manifold $M(F, G, H; g)$ as a skew symmetric tensor field u_{jk} satisfying

$$(2.1)\qquad \nabla_j u_{kl}+\nabla_k u_{jl}=\frac{1}{3}(Q_j{}^t{}_{kl}+Q_k{}^t{}_{jl})\xi_t,$$

where $\xi_t(=\nabla^r u_{rt})$ is the associated vector of u_{kl} and $Q_j{}^t{}_{kl}$ is defined by (1.9), that is,

$$Q_j{}^t{}_{kl}=F_j{}^tF_{kl}+G_j{}^tG_{kl}+H_j{}^tH_{kl}.$$

On the other hand, in a quaternion Kaehlerian manifold $M(F, G, H; g)$ of constant Q-sectional curvature $4k$, it is easily seen that the covariant derivative of a Killing vector ξ_l salisfies the condition [1]

$$(2.2)\qquad \nabla_j\nabla_k\xi_l+\nabla_k\nabla_j\xi_l=-k[2\xi_l g_{jk}-\xi_k g_{jl}-\xi_j g_{kl}+3(Q_j{}^t{}_{kl}+Q_k{}^t{}_{jl})\xi_t].$$

Therefore, for a conformal Killing tensor v_{kl} whose associated vector is ξ_i and a Q-Killing tensor u_{kl} whose associated vector is also ξ_i, if we define a skew symmetric tensor s_{kl} by

$$s_{kl}=v_{kl}+9u_{lk},$$

then we have

$$(2.3)\qquad \nabla_j s_{kl}+\nabla_k s_{jl}=2\xi_l g_{jk}-\xi_k g_{jl}-\xi_j g_{kl}+3(Q_j{}^t{}_{kl}+Q_k{}^t{}_{jl})\xi_t.$$

In this case $-ks_{kl}$ satisfies the condition (2.2). By reason of this fact, we define a *conformal Q-Killing tensor* in a quaternion Kaehlerian manifold as a skew symmetric tensor s_{kl} satisfying (2.3). We call ξ_i the associated vector of s_{kl}. If ξ_i vanishes identically, then s_{kl} is a Killing tensor. From (2.3), we see that the associated vector ξ_i is related to s_{kl} by $\nabla^r s_{ri}=4(n+2)\xi_i$.

Operating ∇_i to (2.3), we get

$$\nabla_i\nabla_j s_{kl}+\nabla_i\nabla_k s_{jl}=2\xi_{il}g_{jk}-\xi_{ik}g_{jl}-\xi_{ij}g_{kl}+3(Q_j{}^t{}_{kl}+Q_k{}^t{}_{jl})\xi_{it}, \tag{2.4}$$

where we have put

$$\xi_{jk}=\nabla_j\xi_k.$$

By interchanging indices i, j, k as $i\to j\to k\to i$ in (2.4), we obtain the following two equations:

$$\nabla_j\nabla_k s_{il}+\nabla_j\nabla_i s_{kl}=2\xi_{jl}g_{ki}-\xi_{ji}g_{kl}-\xi_{jk}g_{il}+3(Q_k{}^t{}_{il}+Q_i{}^t{}_{kl})\xi_{jt}, \tag{2.5}$$

$$\nabla_k\nabla_i s_{jl}+\nabla_k\nabla_j s_{il}=2\xi_{kl}g_{ij}-\xi_{kj}g_{il}-\xi_{ki}g_{jl}+3(Q_i{}^t{}_{jl}+Q_j{}^t{}_{il})\xi_{kt}. \tag{2.6}$$

If we form (2.4)+(2.5)−(2.6), then it follows that

$$\begin{aligned}&2\nabla_i\nabla_j s_{kl}-2K_{kji}{}^t s_{lt}-K_{jkl}{}^t s_{it}-K_{ikl}{}^t s_{jt}-K_{jil}{}^t s_{kt}\\ &=2(\xi_{il}g_{jk}+\xi_{jl}g_{ik}-\xi_{kl}g_{ij})+(\xi_{kj}-\xi_{jk})g_{il}+(\xi_{ki}-\xi_{ik})g_{jl}\\ &\quad-(\xi_{ij}+\xi_{ji})g_{kl}+3(Q_j{}^t{}_{kl}+Q_k{}^t{}_{jl})\xi_{it}+3(Q_k{}^t{}_{il}+Q_i{}^t{}_{kl})\xi_{jt}\\ &\quad-3(Q_i{}^t{}_{jl}+Q_j{}^t{}_{il})\xi_{kt}.\end{aligned} \tag{2.7}$$

Transvecting (2.7) with g^{kl} and taking account of (1.4), we get

$$2K_{jrit}s^{rt}-K_{jirt}s^{rt}+(4n+7)(\xi_{ij}+\xi_{ji})+3(Q_{itjr}+Q_{jtir})\xi^{rt}=0.$$

Taking the symmetric part of this equation, we obtain

$$(4n+7)(\xi_{ij}+\xi_{ji})+3(Q_{itjr}+Q_{jtir})\xi^{rt}=0. \tag{2.8}$$

On the other hand, transvecting (2.8) with $Q_k{}^j{}_h{}^i$, we get

$$9(\xi_{hk}+\xi_{kh})+(4n+13)(Q_{htkr}+Q_{kthr})\xi^{rt}=0. \tag{2.9}$$

Comparing (2.8) with (2.9), we obtain

$$\xi_{ij}+\xi_{ji}=0,\quad (Q_{itjr}+Q_{jtir})\xi^{rt}=0. \tag{2.10}$$

Thus we have the following

THEOREM 1. *In a quaternion Kaehlerian manifold, the associated vector of a conformal Q-Killing tensor is a Killing vector.*

§ 3. Conformal Q-Killing tensor. (continued)

We shall deform (2.7) into another form. Interchanging the indices j, k, l as $j\to k\to l\to j$ in (2.7) and taking account of (2.10), we have the following two equations:

$$
\begin{aligned}
&2\nabla_i\nabla_k s_{lj} - 2K_{lki}{}^t s_{jt} - K_{klj}{}^t s_{it} - K_{ilj}{}^t s_{kt} - K_{kji}{}^t s_{lt} \\
&= 2(\xi_{ij}g_{kl} + \xi_{kj}g_{il} + \xi_{jl}g_{ik} + \xi_{li}g_{kj} + \xi_{lk}g_{ij}) + 3[(Q_k{}^t{}_{lj} + Q_l{}^t{}_{kj})\xi_{it} \\
&\quad + (Q_l{}^t{}_{ij} + Q_i{}^t{}_{lj})\xi_{kt} - (Q_i{}^t{}_{kj} + Q_k{}^t{}_{ij})\xi_{lt}],
\end{aligned} \tag{3.1}
$$

$$
\begin{aligned}
&2\nabla_i\nabla_l s_{jk} - 2K_{jli}{}^t s_{kt} - K_{ljk}{}^t s_{it} - K_{ijk}{}^t s_{lt} - K_{lik}{}^t s_{jt} \\
&= 2(\xi_{ik}g_{lj} + \xi_{lk}g_{ij} + \xi_{kj}g_{il} + \xi_{ji}g_{lk} + \xi_{jl}g_{ik}) + 3[(Q_l{}^t{}_{kj} + Q_k{}^t{}_{lj})\xi_{it} \\
&\quad + (Q_j{}^t{}_{ik} + Q_i{}^t{}_{jk})\xi_{lt} - (Q_i{}^t{}_{lk} + Q_l{}^t{}_{ik})\xi_{jt}].
\end{aligned} \tag{3.2}
$$

Adding (2.7), (3.1) and (3.2) and taking account of (2.3) and (2.10), we get

$$
\begin{aligned}
&2(\nabla_i\nabla_j s_{kl} - \xi_{il}g_{kj} - \xi_{ki}g_{jl}) - (K_{lki}{}^t s_{jt} + K_{jli}{}^t s_{kt} + K_{kji}{}^t s_{lt}) \\
&= (Q_k{}^t{}_{jl} - Q_l{}^t{}_{jk} - 2Q_j{}^t{}_{lk})\xi_{it} + (Q_k{}^t{}_{il} - Q_l{}^t{}_{ik} - 2Q_i{}^t{}_{lk})\xi_{jt} \\
&\quad + (Q_l{}^t{}_{ij} - Q_j{}^t{}_{il} - 2Q_i{}^t{}_{jl})\xi_{kt} + (Q_j{}^t{}_{ik} - Q_k{}^t{}_{ij} - 2Q_i{}^t{}_{kj})\xi_{lt},
\end{aligned} \tag{3.3}
$$

where we have used the relation which follows from (2.3):

$$\nabla_i\nabla_k s_{lj} = \nabla_i\nabla_j s_{kl} - 2\xi_{il}g_{kj} + \xi_{ij}g_{kl} + \xi_{ik}g_{jl} - 3(Q_k{}^t{}_{jl} + Q_j{}^t{}_{kl})\xi_{it}.$$

Subtracting (3.3) from (2.7), we obtain

$$
\begin{aligned}
&K_{jkl}{}^t s_{it} + K_{ilk}{}^t s_{jt} + K_{lij}{}^t s_{kt} + K_{kji}{}^t s_{lt} \\
&= (Q_{lki}{}^t\xi_{jt} - Q_{lkj}{}^t\xi_{it}) + (Q_{lji}{}^t\xi_{kt} - Q_{jlk}{}^t\xi_{it}) \\
&\quad + (Q_{ikj}{}^t\xi_{lt} - Q_{ikl}{}^t\xi_{jt}) + (Q_{ijl}{}^t\xi_{kt} - Q_{ijk}{}^t\xi_{lt}) \\
&\quad + 2(Q_{ilj}{}^t\xi_{kt} - Q_{ilk}{}^t\xi_{jt}) + 2(Q_{jki}{}^t\xi_{lt} - Q_{jkl}{}^t\xi_{it}).
\end{aligned} \tag{3.4}
$$

Transvecting (3.4) with F^{jk} and making use of the formulas (1.5)—(1.7), we obtain after some computations

$$
\begin{aligned}
F_l{}^t\xi_{it} - F_i{}^t\xi_{lt} &= k(F_l{}^t s_{it} - F_i{}^t s_{lt}) + \frac{1}{4n}K_{li}{}^{rk}F_k{}^t s_{rt} \\
&\quad + \frac{k}{n}(\lambda_3 G_{li} - \lambda_2 H_{li}) + \frac{1}{2n}(\mu_3 G_{li} - \mu_2 H_{li}),
\end{aligned} \tag{3.5}
$$

where we have put

$$
\begin{aligned}
&\lambda_1 = F_k{}^t s_t{}^k, \quad \lambda_2 = G_k{}^t s_t{}^k, \quad \lambda_3 = H_k{}^t s_t{}^k, \\
&\mu_1 = F_k{}^t\xi_t{}^k, \quad \mu_2 = G_k{}^t\xi_t{}^k, \quad \mu_3 = H_k{}^t\xi_t{}^k.
\end{aligned} \tag{3.6}
$$

Similarly, transvecting (3.4) with G^{jk} or H^{jk} respectively, we obtain

$$G_l{}^t\xi_{it} - G_i{}^t\xi_{lt} = k(G_l{}^t s_{it} - G_i{}^t s_{lt}) + \frac{1}{4n}K_{li}{}^{rk}G_k{}^t s_{rt}$$

$$(3.7)\qquad +\frac{k}{n}(\lambda_1 H_{li}-\lambda_3 F_{li})+\frac{1}{2n}(\mu_1 H_{li}-\mu_3 F_{li}),$$

$$(3.8)\qquad \begin{aligned} H_l{}^t\xi_{it}-H_i{}^t\xi_{lt}&=k(H_l{}^t s_{it}-H_i{}^t s_{lt})+\frac{1}{4n}K_{li}{}^{rh}H_k{}^t s_{rt}\\ &+\frac{k}{n}(\lambda_2 F_{li}-\lambda_1 G_{li})+\frac{1}{2n}(\mu_2 F_{li}-\mu_1 G_{li}).\end{aligned}$$

By forming $F_{jk}(3.5)+G_{jk}(3.7)+H_{jk}(3.8)$, we find

$$(3.9)\qquad Q_{jkl}{}^t\xi_{it}-Q_{jki}{}^t\xi_{lt}=k(Q_{jkl}{}^t s_{it}-Q_{jki}{}^t s_{lt})+\frac{1}{4n}K_{li}{}^{rh}Q_{jkh}{}^t s_{rt}+A_{jkli},$$

where we have put

$$\begin{aligned} A_{jkli}=&\left(\frac{k}{n}\lambda_1+\frac{1}{2n}\mu_1\right)(G_{jk}H_{li}-H_{jk}G_{li})\\ &+\left(\frac{k}{n}\lambda_2+\frac{1}{2n}\mu_2\right)(H_{jk}F_{li}-F_{jk}H_{li})\\ &+\left(\frac{k}{n}\lambda_3+\frac{1}{2n}\mu_3\right)(F_{jk}G_{li}-G_{jk}F_{li}).\end{aligned}$$

Substituting (3.9) into (3.4), we obtain

$$(3.10)\qquad \begin{aligned} &K_{jkl}{}^t s_{it}+K_{ilk}{}^t s_{jt}+K_{lij}{}^t s_{kt}+K_{kji}{}^t s_{lt}\\ &=k[(Q_{lki}{}^t s_{jt}-Q_{lkj}{}^t s_{it})+(Q_{jli}{}^t s_{kt}-Q_{jlk}{}^t s_{it})\\ &\quad+(Q_{ikj}{}^t s_{lt}-Q_{ikl}{}^t s_{jt})+(Q_{ijl}{}^t s_{kt}-Q_{ijk}{}^t s_{lt})\\ &\quad+2(Q_{ilj}{}^t s_{kt}-Q_{ilk}{}^t s_{jt})+2(Q_{jki}{}^t s_{lt}-Q_{jkl}{}^t s_{it})]\\ &\quad+\frac{1}{4n}T_{lk}{}^r{}_{ij}{}^t s_{rt},\end{aligned}$$

where we have put

$$(3.11)\qquad \begin{aligned} T_{lk}{}^r{}_{ij}{}^t=&Q_{lk}{}^{rh}K_{ijh}{}^t+Q_{jl}{}^{rh}K_{ikh}{}^t+Q_{ik}{}^{rh}K_{jlh}{}^t+Q_{ij}{}^{rh}K_{lkh}{}^t\\ &+2Q_{il}{}^{rh}K_{jkh}{}^t+2Q_{jk}{}^{rh}K_{ilh}{}^t.\end{aligned}$$

We shall write (3.10) as

$$(3.12)\qquad \begin{aligned} &(K_{jkl}{}^t\delta_i{}^r+K_{ilk}{}^t\delta_j{}^r+K_{lij}{}^t\delta_k{}^r+K_{kji}{}^t\delta_l{}^r)s_{rt}\\ &=kD_{lki}{}^t{}_j{}^r s_{rt}+\frac{1}{4n}T_{lk}{}^r{}_{ij}{}^t s_{rt},\end{aligned}$$

where we have put

$$D_{lki}{}^t{}_j{}^r=(Q_{lki}{}^t\delta_j{}^r-Q_{lkj}{}^t\delta_i{}^r)+(Q_{jli}{}^t\delta_k{}^r-Q_{jlk}{}^t\delta_i{}^r)$$

$$+(Q_{ikj}{}^t\delta_l{}^r-Q_{ikl}{}^t\delta_j{}^r)+(Q_{ijl}{}^t\delta_k{}^r-Q_{ijk}{}^t\delta_l{}^r)$$
$$+2(Q_{ilj}{}^t\delta_k{}^r-Q_{ilk}{}^t\delta_j{}^r)+2(Q_{jki}{}^t\delta_l{}^r-Q_{jkl}{}^t\delta_i{}^r).$$

For every point p of the manifold, if $s_{ij}(p)$ takes any preassigned (skew symmetric) value, then (3.12) reduces to

$$\begin{aligned}&K_{jkl}{}^t\delta_i{}^r+K_{ilk}{}^t\delta_j{}^r+K_{lij}{}^t\delta_k{}^r+K_{kji}{}^t\delta_l{}^r\\ &-K_{jkl}{}^r\delta_i{}^t-K_{ilk}{}^r\delta_j{}^t-K_{lij}{}^r\delta_k{}^t-K_{kji}{}^r\delta_l{}^t\\ &=k(D_{lki}{}^t{}_j{}^r-D_{lki}{}^r{}_j{}^t)+\frac{1}{4n}(T_{lk}{}^r{}_{ij}{}^t-T_{lk}{}^t{}_{ij}{}^r).\end{aligned}\tag{3.13}$$

Taking the skew symmetric part of (3.11) with respect to r and t and contracting with respect to r and i, we obtain

$$T_{lk}{}^r{}_{rj}{}^t-T_{lk}{}^t{}_{rj}{}^r=0\tag{3.14}$$

because of

$$Q_{hkr}{}^tK_{jl}{}^{hr}=3K_{jlk}{}^t+8kQ_{jlk}{}^t.$$

Contracting with respect to r and i in (3.13) and taking account of (3.14), we obtain

$$\begin{aligned}&(4n-1)K_{jkl}{}^t-4k(n+2)g_{kl}\delta_j{}^t+4k(n+2)g_{jl}\delta_k{}^t\\ &=k[(1-4n)(Q_{jlk}{}^t-Q_{klj}{}^t-2Q_{kjl}{}^t)+9(-g_{kl}\delta_j{}^t+g_{jl}\delta_k{}^t)],\end{aligned}$$

that is,

$$K_{kjih}=k(g_{kh}g_{ji}-g_{ki}g_{jh}+Q_{khji}-Q_{kijh}-2Q_{kjih}).$$

Thus we obtain the following

THEOREM 2. *If there exists (locally) a conformal Q-Killing tensor which takes any preassigned (skew symmetric) value at any point of a $4n$-dimensional ($n\geqq 2$) quaternion Kaehlerian manifold, then the manifold is of constant Q-sectional curvature.*

§ 4. A sufficient condition for a tensor to be a conformal Q-Killing tensor

Let s_{kl} be a conformal Q-Killing tensor. Then we get

$$\nabla^j\nabla_js_{kl}=4k(n+2)s_{lk}-K_{krlt}s^{rt}-(4n+6)\xi_{kl}-\xi_{lk}+3(Q_{rtkl}+Q_{ktrl})\xi^{rt}\tag{4.1}$$

by transvection (2.7) with g^{ij}.

In this section, we shall show that a skew symmetric tensor s_{kl} satisfying (4.1) is a conformal Q-Killing tensor provided that the manifold is compact and there exists a Q-Killing tensor associated to s_{kl}.

Define a tensor B_{jkl} by

$$B_{jkl}=\nabla_js_{kl}+\nabla_ks_{jl}-2\xi_lg_{kj}+\xi_jg_{kl}+\xi_kg_{jl}-3(Q_{jtkl}+Q_{ktjl})\xi^t\tag{4.2}$$

for a skew symmetric tensor s_{kl}, where ξ_j is given by

$$\nabla^r s_{ri}=4(n+2)\xi_i.$$

Simple computations give us the following equations:

$$\nabla^j B_{jkl}=\nabla^j\nabla_j s_{kl}-4k(n+2)s_{lk}+K_{rklt}s^{rt}+(4n+6)\xi_{kl}+\xi_{lk}-3(Q_{rtkl}+Q_{ktrl})\xi^{rt}, \tag{4.3}$$

$$B_{jkl}B^{jkl}=2B_{jkl}\nabla^j s^{kl}-3B^{rkl}(Q_{rtkl}+Q_{ktrl})\xi^t, \tag{4.4}$$

where we have used the relations $\nabla_j\nabla_k s^{jk}=0$ and

$$\nabla_r\nabla_j s^r{}_k=4(n+2)\xi_{jk}+4k(n+2)s_{jk}-K_{rjkt}s^{rt}.$$

Suppose that there exists a Q-Killing tensor u_{kl} associated with s_{kl} by the condition [2]:

$$\nabla_j u_{kl}+\nabla_k u_{jl}=\frac{1}{3}(Q_{jtkl}+Q_{ktjl})\nabla^r s_r{}^t/4(n+2). \tag{4.5}$$

If we put

$$v_{kl}=-9u_{kl},$$

then by virtue of (4.5), it follows that

$$2B_{jkl}\nabla^j v^{kl}=-3B^{rkl}(Q_{rtkl}+Q_{ktrl})\xi^t. \tag{4.6}$$

Taking account of (4.4) and (4.6) we obtain

$$B_{jkl}B^{jkl}=2B_{jkl}\nabla^j(s^{kl}+v^{kl}). \tag{4.7}$$

Substituting (4.3) and (4.7) into

$$\nabla^j[B_{jkl}(s^{kl}+v^{kl})]=(\nabla^j B_{jkl})(s^{kl}+v^{kl})+B_{jkl}\nabla^j(s^{kl}+v^{kl}),$$

and making use of theorem of Green, we obtain, provided that the manifold is compact,

$$\begin{aligned}\int_M[(s^{kl}+v^{kl})\{\nabla^j\nabla_j s_{kl}-4k(n+2)s_{lk}+K_{rklt}s^{rt}+(4n+6)\xi_{kl}\\ +\xi_{lk}-3(Q_{rtkl}+Q_{ktrl})\xi^{rt}\}+\frac{1}{2}B_{jkl}B^{jkl}]d\sigma=0,\end{aligned} \tag{4.8}$$

where $d\sigma$ denotes the volume element of M and $\nabla_l\nabla^r s_{rk}=4(n+2)\xi_{lk}$.

Thus we have the following

THEOREM 3. *In a compact orientable quaternion Kaehlerian manifold, if the relation* (4.1) *is satisfied for a skew symmetric tensor* s_{kl} *and there exists a Q-Killing tensor* u_{kl} *associated with* s_{kl} *by* (4.5), *then* s_{kl} *is a conformal Q-Killing tensor.*

In section 2, we showed that the covariant derivative of a Killing vector ξ_l in a quaternion Kaehlerian manifold of constant Q-sectional curvature $4k$, satisfies the condition

(2.2).

If we put $\frac{1}{k}\nabla_k\xi_l=v_{kl}$, then for such a tensor v_{kl} the relation (4.6) is also satitfied because of the relations

$$B^r{}_{rk}=0,\quad B^r{}_{kr}=0,\quad B_{kr}{}^r=0.$$

Taking account of theorem 1 and integral formula (4.8), we have the following

THEOREM 4. *In a compact orientable quaternion Kaehlerian manifold of constant Q-sectional curvature, a necessary and sufficient condition for a skew symmetric tensor field s_{kl} to be a conformal Q-Killing tensor is that it satisfies the relation (4.1) and the associated vector of s_{kl} is a Killing vector.*

§ 5. Conformal Q-Killing tensor in a quaternion Kaehlerian manifold of constant Q-sectional curvature.

In this section, we consider a conformal Q-Killing tensor in a quaternion Kaehlerian manifold of constant Q-sectional curvature. For an arbitrary conformal Q-Killing tensor s_{kl}, we put

$$p_{kl}=s_{kl}+(1/k)\nabla_k\xi_l, \tag{5.1}$$

where ξ_l is the associated vector of s_{kl}.

Since the associated vector ξ_l is a Killing vector, the covariant derivative of ξ_l satisfies the condition (2.2). Therefore we easily see that

$$\nabla_j p_{kl}+\nabla_k p_{jl}=0,$$

which means that p_{kl} is a Killing tensor. On the other hand, it is well known that the covariant derivative of a Killing vector is closed. Consequently an arbitrary conformal Q-Killing tensor s_{kl} is decomposed as

$$s_{kl}=p_{kl}+q_{kl},$$

where p_{kl} is a Killing tensor and $q_{kl}=(-1/k)\nabla_k\xi_l$ is a closed conformal Q-Killing tensor. Moreover the uniqueness of this decomposition follows from the following

LEMMA. *In a quaternion Kaehlerian manifold of constant Q-sectional curvature, if a Killing tensor is closed, then it is a zero tensor.*

Proof. Let ξ_{jk} be a closed Killing tensor, then we see easily that $\nabla_l\xi_{jk}=0$ [4]. Thus by virtue of Ricci identity it follows that

$$K_{hlj}{}^t\xi_{tk}+K_{hlk}{}^t\xi_{jt}=0.$$

Transvecting this equation with g^{hj} and taking account of (1.4), we get

$$4k(n+2)\xi_{lk}+K_{lrkt}\xi^{rt}=0. \tag{5.2}$$

Substituting (1.8) into (5.2), we get

(5. 3) $$-(4n+7)\xi_{lk}+(Q_{rtlk}+Q_{lrkt})\xi^{rt}=0,$$

where we have used the relation

$$(Q_{ltkr}+Q_{ktlr})\xi^{rt}=(Q_{krlt}+Q_{ktlr})\xi^{rt}=0.$$

On the other hand, transvecting (5. 3) with $Q_i{}^l{}_j{}^k$, we obtain

(5. 4) $$Q_{rtji}\xi^{rt}=3\xi_{ji}-(4n+5)Q_{jrit}\xi^{rt}.$$

Substituting (5. 4) into (5. 3), we find

(5. 5) $$\xi_{lk}+Q_{lrkt}\xi^{rt}=0.$$

Substituting (5. 5) into (5. 4), we get

$$Q_{rtji}\xi^{rt}=4(n+2)\xi_{ji},$$

and this equation is written as

(5. 6) $$4(n+2)\xi_{ji}+\mu_1 F_{ji}+\mu_2 G_{ji}+\mu_3 H_{ji}=0,$$

where μ_1, μ_2 and μ_3 are defined by (3. 6).

Transvecting (5. 6) with F^{ij}, we easily see that $\mu_1=0$. Similarly we can obtain $\mu_2=0$ and $\mu_3=0$. Therefore we obtain from (5. 6)

$$\xi_{ji}=0.$$

Thus the lemma is proved.

Hence we have the following

THEOREM 5. *In a quaternion Kaehlerian manifold of constant Q-sectional curvature with* $4k$ $(k=K/16n(n+2)\neq 0)$, *a conformal Q-Killing tensor* s_{kl} *is uniquely decomposed as*

$$s_{kl}=p_{kl}+q_{kl},$$

where p_{kl} *is a Killing tensor and* q_{kl} *is a closed conformal Q-Killing tensor. In this case* q_{kl} *is of the form*

$$q_{kl}=(-1/k)\nabla_k\xi_l,$$

where ξ_l *is the associated vector of* s_{kl}.

Conversely if p_{kl} *is a Killing tensor and* ξ_l *is a Killing vector, then* s_{kl} *is a conformal Q-Killing tensor.*

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