J. Korean Math. Soc.
Vol. 13, No. 1, 1976

ANTI-INVARIANT SUBMANIFOLDS OF SASAKIAN SPACE FORMS II

BY KENTARO YANO AND MASAHIRO KON

Introduction

In a previous paper [16] the present authors studied anti-invariant submanifolds of almost contact metric manifolds, especially those of Sasakian manifolds (see also [1], [7], [9], [11] and [12]).

Let $\bar{M}$ be a $(2m+1)$-dimensional almost contact metric manifold with structure tensors $(\phi, \xi, \eta, \bar{g})$. An n-dimensional Riemannian manifold M isometrically immersed in $\bar{M}$ is said to be anti-invariant in $\bar{M}$ if $T_x(M) \perp \phi T_x(M)$ for each $x \in M$, where $T_x(M)$ denotes the tangent space to M at x. If $X \in T_x(M)$, then ϕX is normal to M. Thus we see that, ϕ being of rank $2m$, $n \leqq m+1$. In [16] we studied the case $n=m+1$. In this case the vector field ξ is tangent to M.

The purpose of the present paper is to study n-dimensional anti-invariant submanifolds M normal to the structure vector field ξ of a $(2m+1)$-dimensional almost contact metric manifold $\bar{M}$. When the ambient manifold $\bar{M}$ is a Sasakian manifold, if a submanifold M of $\bar{M}$ is normal to the structure vector field ξ, then, as we shall see from Lemma 1.1, M is anti-invariant in $\bar{M}$. So, in this paper, we mean, by an anti-invariant submanifold M of a Sasakian manifold $\bar{M}$, a submanifold M normal to the structure vector field ξ of a Sasakian manifold $\bar{M}$.

In §1 we recall definitions and some properties of almost contact metric manifolds, especially those of Sasakian manifolds and prove some fundamental properties of anti-invariant submanifolds. In §2 we study n-dimensional anti-invariant submanifolds of a Sasakian space form $\bar{M}(k)$ of dimension $2n+1$ and of constant ϕ-sectional curvature k. Computing the Laplacian of the square of the length of the second fundamental form, we obtain some integral formulas and then using those we prove a pinching theorem, for compact and orientable anti-invariant submanifolds, with respect to the square of the length of the second fundamental form. Moreover we give an example of an anti-invariant surface in S^5. In the last section we study problems for anti-invariant submanifolds of Sasakian space forms similar to those for totally real submanifolds of complex space forms, which we studied in [14].

§1. Anti-invariant submanifolds

Let $\bar{M}$ be a $(2m+1)$-dimensional almost contact metric manifold. We denote by $(\phi, \xi, \eta, \bar{g})$ the structure tensors of $\bar{M}$, where ϕ, ξ, η denote a tensor field of type (1,1), a vector field, a 1-form on $\bar{M}$ respectively, and $\bar{g}$ the Riemannian metric tensor field of $\bar{M}$. Then the structure tensors satisfy the following equations:

$$\phi^2\bar{X} = -\bar{X} + \eta(\bar{X})\xi, \qquad \phi\xi = 0, \qquad \eta(\phi\bar{X}) = 0, \qquad \eta(\xi) = 1,$$

Received by the editors Nov. 7, 1975.

$$g(\phi\bar{X}, \phi\bar{Y})=g(\bar{X}, \bar{Y})-\eta(\bar{X})\eta(\bar{Y}), \quad \eta(\bar{X})=g(\bar{X}, \xi)$$

for any vector fields $\bar{X}$ and $\bar{Y}$ on $\bar{M}$.

An n-dimensional Riemannian manifold M isometrically immersed in a $(2m+1)$-dimensional almost contact metric manifold $\bar{M}$ is called an anti-invariant submanifold if $T_x(M)\perp\phi T_x(M)$ for each $x\in M$ where $T_x(M)$ denotes the tangent space to M at $x\in M$. Here we have identified $T_x(M)$ with its image under the differential of the immersion because our computation is local. By the definition, if X is a tangent vector to M, then ϕX is normal to M. Since the rank of ϕ is $2m$, we have $n-1\leqq(2m+1)-n$, from which $n\leqq m+1$. In [16] we studied the case $n=m+1$. In this case the vector field ξ is tangent to M. We now study anti-invariant submanifolds of an almost contact metric manifold $\bar{M}$ such that the vector field ξ is normal to M.

Let g be the induced metric tensor field of M. We denote by $\bar{\nabla}$ (resp. ∇) the operator of covariant differentiation with respect to $\bar{g}$ (resp. g). Then the Gauss and Weingarten formulas are given respectively by

$$\bar{\nabla}_X Y=\nabla_X Y+B(X, Y) \qquad \text{and} \qquad \bar{\nabla}_X N=-A_N X+D_X N$$

for any vector fields X, Y tangent to M and any vector field N normal to M, where D is the operator of covariant differentiation with respect to the linear connection induced in the normal bundle. We call both A and B the second fundamental forms of M. They satisfy $\bar{g}(B(X, Y), N)=g(A_N X, Y)$.

In the sequel, we assume that the ambient manifold $\bar{M}$ is a $(2m+1)$-dimensional Sasakian manifold. Then we have

$$\bar{\nabla}_{\bar{X}}\xi=\phi\bar{X}, \qquad (\bar{\nabla}_{\bar{X}}\phi)\bar{Y}=-\bar{g}(\bar{X}, \bar{Y})\xi+\eta(\bar{Y})\bar{X}$$

for any vector fields $\bar{X}$ and $\bar{Y}$ on $\bar{M}$.

First of all, we prove

LEMMA 1.1. ([1, 9]). *Let M be an n-dimensional submanifold of a $(2m+1)$-dimensional Sasakian manifold $\bar{M}$. If the vector field ξ is normal to M, then M is an anti-invariant submanifold of $\bar{M}$ and $n\leqq m$.*

Proof. Since the vector field ξ is normal to M, we have

$$\bar{g}(\phi X, Y)=\bar{g}(\bar{\nabla}_X\xi, Y)=g(-A_\xi X, Y)+\bar{g}(D_X\xi, Y)=-g(A_\xi X, Y),$$

for any vector fields X and Y tangent to M. Since A_ξ is symmetric and ϕ is anti-symmetric, we have $A_\xi=0$ and ϕX is normal to M. Thus M is anti-invariant and $n\leqq m$.

Throughout this paper, by an anti-invariant submanifold M of a Sasakian manifold $\bar{M}$, we mean a submanifold M normal to the structure vector field ξ of a Sasakian manifold $\bar{M}$.

In the sequel we study the case $n=m$. For this, we prepare some local formulas for n-dimensional anti-invariant submanifolds of a $(2n+1)$-dimensional Sasakian manifold $\bar{M}$.

We choose a local field of orthonormal frames $e_1, \cdots, e_n$; $e_{0*}=\xi$, $e_{1*}=\phi e_1, \cdots,$

$e_{n*}=\phi e_n$ in $\bar{M}$ in such a way that, restricted to M, $e_1, \cdots, e_n$ are tangent to M and hence $e_{0*}, e_{1*}, \cdots, e_{n*}$ are normal to M. With respect to this frame field of $\bar{M}$, let $w^1, \cdots, w^n$; $w^{0*}, w^{1*}, \cdots, w^{n*}$ be the field of dual frames. Unless otherwise stated we use the conventions that the ranges of indices are respectively:

$$A,\ B,\ C,\ D=1, \cdots, n,\ 0^*, 1^*, \cdots, n^*,$$

$$t,\ s,\ i,\ j,\ k, l=1, \cdots, n,$$

$$a,\ b,\ c,\ d=0^*, 1^*, \cdots, n^*,$$

and that when an index appears twice in any term as a subscript and a superscript, it is understood that this index is summed over its range. Then the structure equations of $\bar{M}$ are given by

$$dw^A=-w^A{}_B\wedge w^B, \quad w^A{}_B+w^B{}_A=0, \tag{1.1}$$

$$dw^A{}_B=-w^A{}_C\wedge w^C{}_B+\Phi^A{}_B, \quad \Phi^A{}_B=\frac{1}{2}K^A{}_{BCD}w^C\wedge w^D. \tag{1.2}$$

When we restrict these forms to M, we have

$$w^a=0. \tag{1.3}$$

Since $0=dw^a=-w^a{}_i\wedge w^i$, by Cartan's lemma, we may write

$$w^a{}_i=h^a{}_{ij}w^j, \quad h^a{}_{ij}=h^a{}_{ji}. \tag{1.4}$$

From these formulas we have

$$dw^i=-w^i{}_j\wedge w^j, \quad w^i{}_j+w^j{}_i=0, \tag{1.5}$$

$$dw^i{}_j=-w^i{}_k\wedge w^k{}_j+\Omega^i{}_j, \quad \Omega^i{}_j=\frac{1}{2}R^i{}_{jkl}w^k\wedge w^l, \tag{1.6}$$

$$R^i{}_{jkl}=K^i{}_{jkl}+\sum_a (h^a{}_{ik}h^a{}_{jl}-h^a{}_{il}h^a{}_{jk}), \tag{1.7}$$

$$dw^a{}_b=-w^a{}_c\wedge w^c{}_b+\Omega^a{}_b, \quad \Omega^a{}_b=\frac{1}{2}R^a{}_{bkl}w^k\wedge w^l, \tag{1.8}$$

$$R^a{}_{bkl}=K^a{}_{bkl}+\sum_i (h^a{}_{ik}h^b{}_{il}-h^a{}_{il}h^b{}_{ik}). \tag{1.9}$$

$$w^i{}_j=w^{i*}{}_{j*}, \quad w^{i*}{}_j=w^{j*}{}_i, \quad w^i=w^{i*}{}_{0*}, \quad w^{i*}=-w^i{}_{0*}. \tag{1.10}$$

From (1.4) and (1.10) we have

$$h^i{}_{jk}=h^j{}_{ik}=h^k{}_{ij}, \quad h^0{}_{ij}=0, \tag{1.11}$$

where we have written $h^0{}_{ij}$, $h^i{}_{jk}$ instead of $h^{0*}{}_{ij}$, $h^{i*}{}_{jk}$ to simplify the notation.

The form $(w^i{}_j)$ defines the Riemannian connection of M. The form $(w^a{}_b)$ defines a connection induced in the normal bundle of M from that of $\bar{M}$. The second fundamental form of M is represented by $h^a{}_{ij}w^iw^je_a$ and is sometimes denoted by its components $h^a{}_{ij}$. If the second fundamental form of M is identically zero, then M is said to be totally geodesic. If the second fundamental form is of the form $\delta_{ij}(\sum_k h^a{}_{kk}e_a)/n$, then M is said to be totally umbilical, where δ_{ij} denotes the Kronecker delta. We call $(\sum_k h^a{}_{kk}e_a)/n$ the mean curvature vector of M and M is said to be minimal if its mean curvature vector vanishes identically, i. e., $\sum_k h^a{}_{kk}=0$ for all a. We define the covariant derivative $h^a{}_{ijk}$ of $h^a{}_{ij}$ by putting

$$h^a{}_{ijk}w^k = dh^a{}_{ij} - h^a{}_{il}w^l{}_j - h^a{}_{lj}w^l{}_i + h^b{}_{ij}w^a{}_b. \tag{1.12}$$

If $h^a{}_{ijk}=0$ for all a, i, j and k, the second fundamental form of M is said to be parallel. If the second fundamental form of M is parallel, then (1.11) and (1.12) imply that $h^o{}_{ijk}=-h^k{}_{ij}=0$, which means that M is totally geodesic. Thus we have

LEMMA 1.2. ([9]). *Let M be an n-dimensional anti-invariant submanifold of a $(2n+1)$-dimensional Sasakian manifold $\bar{M}$. If the second fundamental form of M is parallel, than M is totally geodesic.*

We define the covariant derivative $h^a{}_{ijkl}$ of $h^a{}_{ijk}$ by putting

$$h^a{}_{ijkl}w^l = dh^a{}_{ijk} - h^a{}_{ljk}w^l{}_i - h^a{}_{ilk}w^l{}_j - h^a{}_{ijl}w^l{}_k + h^b{}_{ijk}w^a{}_b. \tag{1.13}$$

Then the Laplacian $\Delta h^a{}_{ij}$ of $h^a{}_{ij}$ is defined by

$$\Delta h^a{}_{ij} = \sum_k h^a{}_{ijkk}. \tag{1.14}$$

By Lemma 1.2, if the second fundamental form of M is parallel, M is totally geodesic. Thus we need the following notion. The second fundamental form of M is said to be η-parallel if $h^t{}_{ijk}=0$ for all t, i, j and k. If the mean curvature vector of M is parallel with respect to the connection in the normal bundle, then the mean curvature vector of M is said to be parallel. If the mean curvature vector of M is parallel, then (1.11) and (1.12) imply that

$$\sum_i h^o{}_{iij} = -\sum_i h^j{}_{ii} = 0 \quad \text{for all } j,$$

which shows that M is minimal. Thus we have

THEOREM 1.1. *Let M be an n-dimensional anti-invariant submanifold of a $(2n+1)$-dimensional Sasakian manifold $\bar{M}$. If the mean curvature vector of M is parallel, then M is minimal.*

From this we see that the notion of parallel mean curvature vector is not essential for anti-invariant submanifolds. Therefore we need the following notion. If $\sum_i h^t{}_{iik}=0$ for all t and k, then we say that M has the η-parallel mean curvature vector.

§2. Anti-invariant submanifolds with parallel mean curvature vector

Let M be an n-dimensional anti-invariant submanifold of a $(2n+1)$-dimensional Sasakian manifold $\bar{M}$. In this section we assume that $\bar{M}$ is of constant ϕ-sectional curvature

k. Then $\bar{M}$ is called a Sasakian space form and is denoted by $\bar{M}^{2n+1}(k)$. In this case we have

$$K^A{}_{BCD}=\frac{1}{4}(k+3)(\delta_{AC}\delta_{BD}-\delta_{AD}\delta_{BC})+\frac{1}{4}(k-1)(\eta_B\eta_C\delta_{AD} -\eta_B\eta_D\delta_{AC}+\eta_A\eta_D\delta_{BC}-\eta_A\eta_C\delta_{BD} +\phi_{AC}\phi_{BD}-\phi_{AD}\phi_{BC}+2\phi_{AB}\phi_{CD}). \tag{2.1}$$

Since the second fundamental form of M satisfies the Codazzi equation $h^a{}_{ijk}=h^a{}_{ikj}$, by a straightforward computation, we have

$$\sum_{a,i,j}h^a{}_{ij}\Delta h^a{}_{ij}=\sum_{a,i,j,k}(h^a{}_{ij}h^a{}_{kkij}+K^l{}_{ijk}h^a{}_{kl}h^a{}_{ij}+K^l{}_{kjk}h^a{}_{li}h^a{}_{ij} -K^a{}_{bjk}h^b{}_{ki}h^a{}_{ij})-\sum_{a,b,i,j,k,l}[(h^a{}_{ik}h^b{}_{jk}-h^a{}_{jk}h^b{}_{ik})(h^a{}_{il}h^b{}_{jl}-h^a{}_{jl}h^b{}_{il}) +h^a{}_{ij}h^a{}_{kl}h^b{}_{ij}h^b{}_{kl}-h^a{}_{ij}h^a{}_{ki}h^b{}_{kj}h^b{}_{ll}].$$

Here we notice that $g(A_a e_i, e_j)=h^a{}_{ij}$ where $A_a=A_{e_a}$. Thus A_a is represented in a matrix form $A_a=(h^a{}_{ij})$, which is a symmetric (n,n)-matrix. Thus we have

$$\sum_{a,i,j}h^a{}_{ij}\Delta h^a{}_{ij}=\sum_{a,i,j,k}h^a{}_{ij}h^a{}_{kkij}+\sum_a\left\{\frac{1}{4}(n(k+3)+(k-1))\mathrm{Tr}A_a{}^2 -\frac{1}{2}(k+1)(\mathrm{Tr}A_a)^2\right\}+\sum_{a,b}\{\mathrm{Tr}(A_aA_b-A_bA_a)^2 -[\mathrm{Tr}(A_aA_b)]^2+\mathrm{Tr}A_b\mathrm{Tr}(A_aA_bA_a)\}. \tag{2.2}$$

Now we put

$$S_{ab}=\sum_{i,j}h^a{}_{ij}h^b{}_{ij},\quad S_a=S_{aa},\quad S=\sum_a S_a,$$

then S is the square of the length of the second fundamental form. From (1.11) and (1.12) we obtain

$$\sum_{a,i,j}h^a{}_{ij}\Delta h^a{}_{ij}=\frac{1}{2}\Delta S-\sum_{a,i,j,k}(h^a{}_{ijk})^2 =\frac{1}{2}\Delta S-\sum_{t,i,j,k}(h^t{}_{ijk})^2-S. \tag{2.3}$$

From (1.11) we see that $S_{0b}=A_0=0$ and (2..2) and (2.3) imply the following

LEMMA 2.1. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$. Then we have*

$$\frac{1}{2}\Delta S-\sum_{t,i,j,k}(h^t{}_{ijk})^2=\sum_{t,i,j,k}h^t{}_{ij}h^t{}_{kkij}+\frac{1}{4}(n+1)(k+3)S \tag{2.4}$$

$$-\frac{1}{2}(k+1)\sum_t(\mathrm{Tr}A_t)^2+\sum_{t,s}\{\mathrm{Tr}(A_tA_s-A_sA_t)^2$$

$$-[\mathrm{Tr}(A_tA_s)]^2+\mathrm{Tr}A_s\mathrm{Tr}(A_tA_sA_t)\}.$$

Since S_{ts} is a symmetric (n,n)-matrix, we can assume that S_{ts} is diagonal by a suitable choice of $e_{1*},\cdots,e_{n*}$. If the mean curvature vector of M is parallel, by Theorem 1.1, equation (2.4) can be rewritten as (2.5) below. Thus we have

LEMMA 2.2. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$. If the mean curvature vector of M is parallel, then*

$$\frac{1}{2}\Delta S-\sum_{t,i,j,k}(h^t{}_{ijk})^2=\frac{1}{4}(n+1)(k+3)S-\sum_t S_t^2 \tag{2.5}$$

$$+\sum_{t,s}\mathrm{Tr}(A_tA_s-A_sA_t)^2.$$

In the sequel, we need the following lemma.

LEMMA 2.3. ([3]). *Let A and B be symmetric (n,n)-matrices. Then*

$$-\mathrm{Tr}(AB-BA)^2\leqq 2\mathrm{Tr}A^2\mathrm{Tr}B^2,$$

and the equality holds for non-zero matrices A and B if and only if A and B can be transformed simultaneously by an orthogonal matrix into scalar multiples of $\tilde{A}$ and $\tilde{B}$ respectively, where

$$\tilde{A}=\left(\begin{array}{cc|c} 0 & 1 & \\ 1 & 0 & 0 \\ \hline & 0 & 0\end{array}\right),\quad \tilde{B}=\left(\begin{array}{cc|c} 1 & 0 & \\ 0 & -1 & 0 \\ \hline & 0 & 0\end{array}\right).$$

Moreover, if A_1, A_2, A_3 are symmetric (n,n)-matrices such that

$$-\mathrm{Tr}(A_aA_b-A_bA_a)^2=2\mathrm{Tr}A_a^2\mathrm{Tr}A_b^2,\quad 1\leqq a,b\leqq 3,\quad a\neq b,$$

then at least one of the matrices A_a must be zero.

From Lemma 2.3, we have

$$-\sum_{t,s}\mathrm{Tr}(A_tA_s-A_sA_t)^2+\sum_t S_t^2-\frac{1}{4}(n+1)(k+3)S \tag{2.6}$$

$$\leqq 2\sum_{t\neq s}S_tS_s+\sum_t S_t^2-\frac{1}{4}(n+1)(k+3)S$$

$$=[(2-\frac{1}{n})S-\frac{1}{4}(n+1)(k+3)]S-\frac{1}{n}\sum_{t>s}(S_t-S_s)^2.$$

From (2.5) and (2.6) we have

$$\sum_{t,i,j,k}(h^t{}_{ijk})^2-\frac{1}{2}\Delta S\leqq[(2-\frac{1}{n})S-\frac{1}{4}(n+1)(k+3)]S. \tag{2.7}$$

If M is compact and orientable, (2. 7) implies the following

THEOREM 2. 1. *Let M be an n-dimensional compact orientable anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$. If the mean curvature vector of M is parallel, then*

$$\int_M [(2-\frac{1}{n})S-\frac{1}{4}(n+1)(k+3)]S^*1 \geqq \int_M \sum_{t,i,j,k}(h^t{}_{ijk})^{2}{}^*1 \geqq 0. \tag{2.8}$$

COROLLARY 2. 1. *Let M be an n-dimensional compact orientable anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$ with parallel mean curvature vector. Then either M is totally geodesic, or $S=n(n+1)(k+3)/4(2n-1)$, or at some point $x\in M$, $S(x)>n(n+1)(k+3)/4(2n-1)$.*

Proof. Suppose $S\leqq n(n+1)(k+3)/4(2n-1)$ everywhere on M. Then (2. 8) shows that the second fundamental form of M is η-parallel and hence S is a constant. Thus either $S=0$, i. e., M is totally geodesic or $S=n(n+1)(k+3)/4(2n-1)$. Except for these possibilities, $S(x)>n(n+1)(k+3)/4(2n-1)$ at some $x\in M$.

THEOREM 2. 2. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$ $(n>1)$ with parallel mean curvature vector. If $k=1$, that is, M is of constant curvature 1, and if $S=n(n+1)/(2n-1)$, then $n=2$ and M is a flat surface of $M^5(1)$. With respect to an adapted dual orthonormal frame field w^1, w^2, $w^{0*}=\eta$, w^{1*}, w^{2*}, the connection form $(w^A{}_B)$ of $M^5(1)$, restricted to M, is given by*

$$\begin{pmatrix} 0 & 0 & 0 & -\lambda w^2 & -\lambda w^1 \\ 0 & 0 & 0 & -\lambda w^1 & \lambda w^2 \\ 0 & 0 & 0 & -w^1 & -w^2 \\ \lambda w^2 & \lambda w^1 & w^1 & 0 & 0 \\ \lambda w^1 & -\lambda w^2 & w^2 & 0 & 0 \end{pmatrix}, \qquad \lambda=\frac{1}{\sqrt{2}}.$$

Proof. From the assumption and Theorem 1. 1, M is minimal. Since S is a constant, by (2. 7), we see that the second fundamental form of M is η-parallel, i. e., $h^t{}_{ijk}=0$. From (2. 6) and (2. 7) we have

$$\sum_{t>s}(S_t-S_s)^2=0, \tag{2.9}$$

$$-\mathrm{Tr}(A_tA_s-A_sA_t)^2=2\mathrm{Tr}A_t^2\mathrm{Tr}A_s^2 \tag{2.10}$$

for any t and s, and hence $S_t=S_s$ for all t and s and we may assume that $A_t=0$ for $t=3,\cdots,n$. Therefore we must have $n=2$ and we obtain

$$A_0=0, \qquad A_1=\lambda\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \qquad A_2=\lambda\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{2.11}$$

by using $h^1{}_{12}=\lambda=h^2{}_{11}$. Since $h^t{}_{ijk}=0$, (1. 12) implies that

$$dh^t{}_{ij}=h^t{}_{ik}w^k{}_j+h^t{}_{kj}w^k{}_i-h^k{}_{ij}w^{t*}{}_{k*}. \tag{2.12}$$

Putting $t=1$, $i=1$ and $j=2$, we have $d\lambda=0$, which means that λ is a constant. Since $S=2$, we have $2\lambda^2=1$. Thus we may assume that $\lambda=1/\sqrt{2}$. Moreover (1.4) and (2.11) imply the following:

$$w^{1*}{}_1=\lambda w^2, \quad w^{1*}{}_2=w^{2*}{}_1=\lambda w^1, \quad w^{2*}{}_2=-\lambda w^2.$$

On the other hand, by (1.10) and (2.12), we have $0=dh^1{}_{11}=3\lambda w^2{}_1$. Since $\lambda\neq 0$, we see that

$$w^2{}_1=w^{2*}{}_{1*}=0.$$

From (1.10) we also have

$$w^{1*}{}_{0*}=w^1, \quad w^{2*}{}_{0*}=w^2, \quad w^1{}_{0*}=w^2{}_{0*}=0.$$

From the Gauss equation (1.7) and (2.11) we easily see that M is flat. From these considerations we have our assertion.

EXAMPLE. Let J be the almost complex structure of $C^3=E^6$ given by

$$J=\begin{pmatrix} 0 & -1 & & & & \\ 1 & 0 & & & & \\ & & 0 & -1 & & \\ & & 1 & 0 & & \\ & & & & 0 & -1 \\ & & & & 1 & 0 \end{pmatrix}.$$

Let S^5 be a 5-dimensional unit sphere in C^3. Then S^5 has the standard Sasakian structure. For any $z\in S^5$, put $\xi=Jz$, and consider the orthogonal projection

$$\pi : T_z(C^3)\longrightarrow T_z(S^5).$$

Putting $\phi=\pi\circ J$, we have a Sasakian structure $(\phi, \xi, \eta, \bar{g})$ on S^5, where η is a 1-form dual to ξ and $\bar{g}$ the standard metric tensor field on S^5.

Let $T=S^1\times S^1$ be a torus. Then we can construct an isometric minimal immersion of T into S^5 which is anti-invariant. Let X: $T\longrightarrow S^5$ be a minimal immersion represented by

$$X=\frac{1}{\sqrt{3}}(\cos\theta, \ \sin\theta, \ \cos\tau, \ \sin\tau, \ \cos\gamma, \ \sin\gamma),$$

where we have put $\gamma=-(\theta+\tau)$. Then we may consider X as a position vector in S^5. The vector field ξ on S^5 is given by

$$\xi=JX=\frac{1}{\sqrt{3}}(-\sin\theta, \ \cos\theta, \ -\sin\tau, \ \cos\tau, \ -\sin\gamma, \ \cos\gamma).$$

Moreover we have

$$X_\theta=\partial X/\partial\theta=\frac{1}{\sqrt{3}}(-\sin\theta,\ \cos\theta,\ 0,\ 0,\ \sin\gamma,\ -\cos\gamma),$$

$$X_\tau=\partial X/\partial\tau,=\frac{1}{\sqrt{3}}(0,\ 0,\ -\sin\tau,\ \cos\tau,\ \sin\gamma,\ -\cos\gamma).$$

Thus X_θ and X_τ are linearly independent and $\eta(X_\theta)=\eta(X_\tau)=0$, whcih shows that the immersion X is anti-invaiarnt. Let

$$M^3=S^1\left(\frac{1}{\sqrt{3}}\right)\times S^1\left(\frac{1}{\sqrt{3}}\right)\times S^1\left(\frac{1}{\sqrt{3}}\right).$$

Then M^3 is an anti-invariant submanifold tangent to the structure vector field ξ of S^5. By the definition of the immersion X, T is immersed in M^3 as a totally geodesic surface. Thus T is an anti-invariant flat minimal surface of S^5 by the immersion X. Moreover we see that the square of the length of the second fundamental form of T in S^5 is 2, that is, $S=2$.

From these considerations and Theorem 2.2, we have the following

THEOREM 2.3. *Let M be an n-dimensional compact orientable anti-invariant submanifold of S^{2n+1} $(n>1)$ with parallel mean curvature vector. If $S=n(n+1)/(2n-1)$, then M is $S^1\times S^1$.*

§3. Anti-invariant submanifolds with constant curvature

Let M be an n-dimensional anti-invariant submanifold of a $(2n+1)$-dimensional Sasakian manifold $\bar{M}$. If $A_aA_b=A_bA_a$ for all a and b, then the second fundamental form of M is said to be commutative, which is equivalent to that $\sum_j h^a{}_{ij}h^b{}_{jk}=\sum_j h^a{}_{jk}h^b{}_{ij}$ for all a, b, i and k.

LEMMA 3.1. *Let M be an n-dimensional anti-invariant submanifold of a $(2n+1)$-dimensional Sasakian manifold $\bar{M}$. Then the second fundamental form of M is commutative if and only if we can choose an orthonormal frame for which $h^t{}_{ij}=0$ unless $t=i=j$, i.e., $A_t=(h^t{}_{ij})$ is of the form*

$$A_t=\begin{pmatrix} 0 & & & & & \\ & \ddots & & & & \\ & & 0 & & & \\ & & & \lambda_t & & \\ & & & & 0 & \\ & & & & & \ddots \\ & & & & & & 0\end{pmatrix} t, \qquad t=1,\cdots,n.$$

Proof. Assume that the second fundamental form of M is commutative, that is, $A_aA_b=A_bA_a$, $a,b=n+1,\cdots,2n+1$. Then we can choose an orthonormal frame $e_1,\cdots,e_n$ for

$T_x(M)$ in such a way that all A_a are simultaneously diagonal, i.e., $h^a{}_{ij}=0$ when $i\neq j$, that is, $h^t{}_{ij}=0$ when $i\neq j$. From (1.11) we see that $h^t{}_{ij}=0$ unless $t=i=j$. It is easy to see that the converse is also true.

COROLLARY 3.1. *Let M be an n-dimensional anti-invariant submanifold of a $(2n+1)$-dimensional Sasakian manifold $\bar{M}$ with parallel mean curvature vector. If the second fundamental form of M is commutative, then M is totally geodesic.*

Proof. From Theorem 1.1, M is minimal. Thus we have $\mathrm{Tr}A_t=0$ for all t, and hence $\lambda_t=0$ by Lemma 3.1. Since $A_0=0$, M is totally geodesic.

COROLLARY 3.2. *Let M be an n-dimensional $(n>1)$ anti-invariant submanifold of a $(2n+1)$-dimensional Sasakian manifold $\bar{M}$. If M is totally umbilical, then M is totally geodesic.*

Proof. Since M is umbilical, we have $h^t{}_{ij}=\delta_{ij}(\mathrm{Tr}A_t)/n$ and $A_0=0$. Therefore the second fundamental form of M is commutative. Thus Lemma 3.1 implies that $h^t{}_{ij}=0$ unless $t=i=j$. On the other hand, we have $h^t{}_{ij}=\lambda_t\delta_{ij}/n$. Putting $i=j\neq t$, we have $\lambda_t=0$ and hence M is totally geodesic.

Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$. Then (1.7) and (2.1) imply that

$$R^i{}_{jkl}=\tfrac{1}{4}(k+3)(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+\sum_a(h^a{}_{ik}h^a{}_{jl}-h^a{}_{il}h^a{}_{jk}). \tag{3.1}$$

If M is totally geodesic, then M is of constant curvature $\frac{1}{4}(k+3)$. From this, Corollary 3.1 and 3.2 we have

COROLLARY 3.3. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$ with parallel mean curvature vector. If the second fundamental form of M is commutative, then M is of constant curvature $\frac{1}{4}(k+3)$.*

COROLLARY 3.4. *Let M be an n-dimensional $(n>1)$ anti-invariant and totally umbilical submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$. Then M is of constant curvature $\frac{1}{4}(k+3)$.*

LEMMA 3.2. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$. Then M is of constant curvature $\frac{1}{4}(k+3)$ if and only if M has the commutative second fundamental form.*

Proof. From (1.7), (1.11) and (3.1) we have

$$\begin{aligned} R^i{}_{jkl}&=K^i{}_{jkl}+\sum_t(h^t{}_{ik}h^t{}_{jl}-h^t{}_{il}h^t{}_{jk}) \\ &=\tfrac{1}{4}(k+3)(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+\sum_t(h^i{}_{tk}h^j{}_{tl}-h^i{}_{tl}h^j{}_{tk}), \end{aligned} \tag{3.2}$$

which proves our assertion.

LEMMA 3.3. *Let M be an n-dimensional anti-invariant submanifold of a $(2n+1)$-dimensional Sasakian manifold $\tilde{M}$. Then*

$$\sum_{t,s}\mathrm{Tr}A_t^2A_s^2=\sum_{t,s}(\mathrm{Tr}A_tA_s)^2. \tag{3.3}$$

Proof. Since $h^i{}_{jk}=h^j{}_{ik}=h^k{}_{ij}$, we have

$$\sum_{t,s}\mathrm{Tr}A_t^2A_s^2=\sum_{t,s,i,j,k,l} h^t{}_{kl}h^t{}_{li}h^s{}_{ij}h^s{}_{jk}$$

$$=\sum_{t,s,i,j,k,l} h^k{}_{tl}h^i{}_{lt}h^i{}_{sj}h^k{}_{js}=\sum_{k,i}(\mathrm{Tr}A_kA_i)^2.$$

LEMMA 3.4. *Let M be an n-dimensional anti-invariant submanifold with constant curvature c of a Sasakian space form $\tilde{M}^{2n+1}(k)$. Then we have*

$$[\tfrac{1}{4}(k+3)-c]\sum_t[\mathrm{Tr}A_t^2-(\mathrm{Tr}A_t)^2]=\sum_{t,s}[\mathrm{Tr}A_t^2A_s^2-\mathrm{Tr}(A_tA_s)^2]. \tag{3.4}$$

Proof. From the assumption and (3.2) we have

$$[\tfrac{1}{4}(k+3)-c](\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})=\sum_t(h^t{}_{il}h^t{}_{jk}-h^t{}_{ik}h^t{}_{jl}). \tag{3.5}$$

Multiplying both sides of (3.5) by $\sum_s h^s{}_{il}h^s{}_{jk}$ and summing up with respect to i, j, k and l, we have (3.4) by (3.3).

LEMMA 3.5. *Let M be an n-dimensional anti-invariant submanifold with constant curvature c of a Sasakian space form $\tilde{M}^{2n+1}(k)$. Then we have*

$$(n-1)[\tfrac{1}{4}(k+3)-c]\sum_t\mathrm{Tr}A_t^2=\sum_{t,s}[\mathrm{Tr}A_t^2A_s^2-\mathrm{Tr}A_s\mathrm{Tr}(A_tA_sA_t)]. \tag{3.6}$$

Proof. From (3.5) we obtain

$$(n-1)[\tfrac{1}{4}(k+3)-c]\delta_{jl}=\sum_{t,i}(h^t{}_{il}h^t{}_{ij}-h^t{}_{ii}h^t{}_{jl}). \tag{3.7}$$

Multiplying both sides of (3.7) by $\sum_s h^s{}_{jk}h^s{}_{kl}$ and summing up with respect to i, k and l we have (3.6).

LEMMA 3.6. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\tilde{M}^{2n+1}(k)$ with η-parallel mean curvature vector. If the scalar curvature R of M is constant, then the square of the length of the second fundamental form is constant, i.e., $S=\sum_t\mathrm{Tr}A_t^2=consant$.*

Proof. From (3.2) we have

$$R=\tfrac{1}{4}n(n-1)(k+3)+\sum_t(\mathrm{Tr}A_t)^2-\mathrm{S},$$

from which we have our assertion since R and $\sum_t(\mathrm{Tr}A_t)^2$ are both constant by the assumption.

LEMMA 3.7. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$ with η-parallel mean curvature vector. If M is of constant curvature c, then*

$$\sum_{t,i,j,k}(h^t{}_{ijk})^2=-c\sum_t[(n+1)\mathrm{Tr}A_t^2-2(\mathrm{Tr}A_t)^2]. \tag{3.8}$$

Proof. Since M has η-parallel mean curvature vector, by (1.13) we have $-\sum_{t,i,j,k}h^t{}_{ij}h^t{}_{kkij}=\sum_t(\mathrm{Tr}A_t)^2$. From this and Lemma 3.6, equation (2.4) becomes

$$\sum_{t,i,j,k}(h^t{}_{ijk})^2=-\sum_t[\tfrac{1}{4}(n+1)(k+3)\mathrm{Tr}A_t^2-\tfrac{1}{2}(k+3)(\mathrm{Tr}A_t)^2] \tag{3.9}$$

$$-\sum_{t,s}\{\mathrm{Tr}(A_tA_s-A_sA_t)^2-[\mathrm{Tr}(A_tA_s)]^2+\mathrm{Tr}A_s\mathrm{Tr}(A_tA_sA_t)\}.$$

Substituting (3.4) and (3.6) into (3.9) and using (3.3), we have (3.8).

Now we have

PROPOSITION 3.1. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$ with η-parallel mean curvature vector. If M is flat, then the second fundamental form of M is η-parallel.*

Proof. If M is flat, then by Lemma 3.7, we see that $h^t{}_{ijk}=0$ and hence the second fundamental form of M is η-parallel.

THEOREM 3.1. *Let M be an n-dimensional $(n>1)$ anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$ with η-parallel mean curvature vector and of constant curvature c. If $\frac{1}{4}(k+3)\geqq c$, then either $c\leqq 0$ or M is totally geodesic and hence $\frac{1}{4}(k+3)=c$.*

Proof. From (3.7) we have

$$[\tfrac{1}{4}(k+3)-c]n(n-1)=\sum_t[\mathrm{Tr}A_t^2-(\mathrm{Tr}A_t)^2].$$

By the assumption we obtain

$$\sum_t\mathrm{Tr}A_t^2\geqq\sum_t(\mathrm{Tr}A_t)^2. \tag{3.10}$$

If $c>0$, then (3.8) implies that

$$0=\sum_t\{(n-1)\mathrm{Tr}A_t^2+2[\mathrm{Tr}A_t^2-(\mathrm{Tr}A_t)^2]\},$$

which shows that $\sum_t\mathrm{Tr}A_t^2=0$ and hence M is totally geodesic. Except for this possibility, we have $c\leqq 0$.

THEOREM 3.2. *Let M be an n-dimensional $(n>1)$ anti-invarient submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$ with η-parallel second fundamental form and of constant curvature c. If $\frac{1}{4}(k+3)\geqq c$, then either M is totally geodesic or flat, that is, $c=0$.*

LEMMA 3.8. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$ with parallel mean curvature vector and of constant curvature c. Then we have*

$$\sum_{t,i,j,k}(h^t{}_{ijk})^2=-c(n+1)S. \tag{3.11}$$

Proof. By the assumption and Theorem 1.1, M is minimal and hence (3.8) becomes (3.11).

COROLLARY 3.5. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$ with parallel mean curvature vector and of constant curvature c. Then either $c\leqq 0$ or M is totally geodesic.*

COROLLARY 3.6. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$ with parallel mean curvature vector and of constant curvature c. If the second fundamental form of M is η-parallel, then either M is flat or totally geodesic.*

LEMMA 3.9. *Let M be an n-dimensional anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$ with η-parallel mean curvature vector. If the second fundamental form of M is commutative, then we have*

$$\sum_{t,i,j,k}(h^t{}_{ijk})^2=-\tfrac{1}{4}(k+3)(n-1)S. \tag{3.12}$$

Proof. Using Lemma 3.1 and Lemma 3.2, we can transform (3.8) into (3.12).

THEOREM 3.3. *Let M be an n-dimensional $(n>1)$ anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$ with η-parallel mean curvature vector and with commutative second fundamental form. Then either M is totally geodesic or $k\leqq -3$.*

THEOREM 3.4. *Let M be an n-dimensional $(n>1)$ anti-invariant submanifold of a Sasakian space form $\bar{M}^{2n+1}(k)$ with η-parallel second fundamental form. If the second fundamental form of M is commutative, then M is either totally geodesic or flat.*

Proof. By the assumption and Lemma 3.2, M is of constant curvature $\frac{1}{4}(k+3)$. On the other hand, by (3.12), M is either totally geodesic or $k=-3$ in which case M is flat.

References

[1] D. E. Blair and K. Ogiue, *Geometry of integral submanifolds of a contact distribution,* Illinois J. Math., **19** (1975), 269—276.

[2] B. Y. Chen and K. Ogiue, *On totally real submanifolds,* Trans. Amer. Math. Soc., **193** (1974), 257—266.

[3] S. S. Chern, M. do Carmo and S. Kobayashi, *Minimal submanifolds of a sphere with second fundamental form of constant length,* Functional analysis and related fields, Springer, (1970), 59—75.

[4] M. Kon, *Totally real submanifolds in a Kaehler manifolds*, to appear in J. Differential Geometry.

[5] G. D. Ludden, M. Okumura and K. Yano, *A totally real surface in CP^2 that is not totally geodesic*, Proc. Amer. Math. Soc., **53** (1975), 186—190.

[6] G. D. Ludden, M. Okumura and K. Yano, *Totally real submanifolds of complex manifolds*, Rendicont: dell' Accademia Naziondle dei Lincei, **58** (1975), 346—353.

[7] G. D. Ludden, M. Okumura and K. Yano, *Anti-invariant submanifolds of almost contact metric manifolds*, to appear.

[8] S. Sasaki, *Almost contact manifolds*, Lecture Note I, Tohoku University, (1965).

[9] S. Yamaguchi, M. Kon and T. Ikawa, *On C-totally real submanifolds*, to appear in J. Differential Geometry.

[10] K. Yano, *Differential geometry of totally real submanifolds*, to appear in J. Differential Geometry.

[11] K. Yano, *Anti-invariant submanifolds of a Sasakian manifold with vanishing contact Bochner curvature tensor*, to appear.

[12] K. Yano, *Differential geometry of anti-invariant submanifolds of a Sasakian manifold*, to appear.

[13] K. Yano and S. Ishihara, *Submanifolds with parallel mean curvature vector*, J. Differential Geometry, **6** (1971), 95—118.

[14] K. Yano and M. Kon, *Totally real submanifolds of complex space forms*, to appear in Tôhoku Math. J..

[15] K. Yano and M. Kon, *Totally real submanifolds of complex space forms* II, to appear in Kōdai Math. Sem. Rep..

[16] K. Yano and M. Kon, *Anti-invariant submanifolds of Sasakian space forms* I, to appear in Tôhoku Math. J..

Tokyo Institute of Technology,
Science University of Tokyo.