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A NOTE ON RINGS WHOSE QUASI-INJECTIVE MODULES ARE INJECTIVE

By

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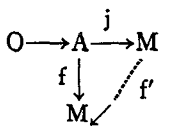
1. Introduction

Throughout this paper, we shall assume that every ring has an identity and that every module is unitary. Let R be a ring. An R-module M is called quasi-injective if for every R-module A and R-homomorphism $f: A \longrightarrow M$, R-monomorphism $j: A \longrightarrow M$, there is an $f' \in End_R(M)$ such that the diagram commutes.



A ring R is called a right (left) qc-ring if every cyclic right (left) R-module M is quasi-injective. This paper is concerned with qc-rings and other rings. This paper investigates the connection between QII-rings and rings whose semisimple right R-modules are injective (SSI-ring). We shall show that a V-ring is an SSI-ring if and only if it is noetherian. We also show that a regular ring R is an SSI-ring if and only if it is a QII-ring. In §3 we will introduce some properties for a qc-rings. We shall show that a right qc-ring is a QII-ring if and only if it is a V-ring.

We also show that a right duo ring is a right qc-ring if and only if each factor ring is self-injective. We will show that a V-ring is a right qc-ring if and only if R_n is a right qc-ring.

2. V-ring and SSI-ring

A ring R is called a right V-ring if and only if every simple right R-module is injective. R is called an SSI-ring if every semi-simple right R-module is injective. The following theorem is generally attributed to Villamayor.

Theorem. The following are equivalent:

(a) R is a V-ring.

(b) Every right ideal of R is the is the intersection of the maximal right ideals which contain it.

Proof. (a) implies (b). Let I be a nonzero right R-module. If $0 \neq x \notin I$, let Kx be a maximal along the right ideals of R which exclude x. R/Kx is uniform since every nonzero submodule contains the right R-module $(xR+Kx)/Kx$ generated by $x+Kx$. Since $(xR+Kx)/Kx$ is simple, it is injective and a direct summand of R/Kx. Thus $R/Kx=(xR+Kx)/Kx$ and Kx is a maximal right ideal of R.

$x \notin$ Rad R/I. (b) implies (a). Let S be simple, I a right ideal oi R and ϕ an R-homomorphism of I into S. We need to extend ϕ to a mapping of R into S. We may assume ϕ is epi and replace ϕ by the cannonical epimorphism $\pi' : I \longrightarrow I/K$, $\pi'(i)=i+K$, where $K=\mathrm{Ker}\,\phi$, $I/K \cong S$. It suffices to extend π' to R. (b) implies Rad $R/K=0$ and since I/K is simple there is a maximal submodule M/K of R/K such that $I/K \oplus M/K=R/K$. The canonical epimorphism $\pi : R \longrightarrow R/K$ composed with the projection onto I/K then extends π'.

Proposition 1. Let R be a V-ring. R is an SSI-ring if and ooly if R_R is noetherian.

Proof. Let M be a semisimple right R-module. Assume that R_R is noetherian. M is a direct sum of simple modules. Since R is a V-ring, every simple module is injective. Let $M=\sum_{j\in J}\oplus M_j$ be a direct sum of a family $\{M_j \mid j \in J\}$ of simple modules. If I is any right ideal of R, then there exist $a_1, \cdots, a_n \in R$ such that $I=\sum_1^n a_iR$.

If f is any map of I in M, then there is a finite subset P of J such that $f(a_i) \in N=\sum_{p\in P} M_p$ $i=1, \cdots, n$, and then $f(I) \subseteq N$. Since N is injective, by Baer's criterion there exists $m \in N$ such that $f(x)=mx$ for all $x \in I$. M therefore satisfies Baer's condition and is accordingly injective. If $0=I_1 \subset I_2 \subset \cdots \subset I_n \cdots$ is an ascending chain of right ideals of R, let I denote their union.

Since R is a V-ring we can find right ideals M_k such that $I_{k-1} \subseteq M_k \subset I_k$ and I_k/M_k is simple. The canonical epimorphism $I \rightarrow I/M_k$ together with the projections $I/M_k \rightarrow I_k/M_k$ compose to give epimorphism $\pi_k : I \rightarrow I/M$ such that if $x \in I_j$ then $\pi_k(x)=0$ for $k>j$. Thus the π_k's induce a map, $\pi : I \rightarrow \coprod I_k/M_k$. We see that Im π is contained in $\coprod^n_{k=1} I_k/M_k$ for some integer n Since $\pi_k(x) \neq 0$: for $x \in I_k - M_k$ we see that our sequence has length $\leqq n$.

A ring R is called a QII-ring if every right quasi-injective module is injective.

Proposition 2. Let R be a regular ring. Then a ring R is an SII-ring if and only if R is a QII-ring.

Proof. Let R be an SSI-ring. By proposition 1 R is noetherian.

Thus R is regular and noetherian [2]. Thus R is semi-simple and artinian [3]. Therefore R is a QII-ring. Conversely assume that R is a QII-ring. Since every semi-simple module is quasi-injective, R is an SSI-ring.

3. qc-rings and other rings

A ring R is called a right(left) qc-ring if every cyclic right(left) R-module is quasi-injective. A ring R is called semi-perfect if every finitely generated right R-module has a projective cover.

Theorem 1. Let R be a right qc-ring. Then the following statements are equivalent:

(1) R is a V-ring.

(2) R is an SSI-ring.

(3) R is a QII-ring.

Proof. (1) implies (2). Since R_R is generated by the identity, R_R is a cyclic right R-module. R_R is quasi-injective. Then any homomorphism from a right ideal of R into R_R can be extended to a homomorphism of R into R. Hence R_R is injective by Baer's criterion. In other words, R is right self-injective. Since R is a V-ring, the Jacobson radical of R is zero. We shall first prove that R is semi-perfect. ThusR is regular in the sense of Von-neuman. We will show that R does not contain an infinite set of non-zero orthogonal idempotents. Let us suppose that R contains an infinite set $\{e_i | i \in I\}$ of nonzero orthogonal idempotents. Since R is right self-injective and regular, for each $A \subseteqq I$ there exists an idempotent $m_A \in R$ such that $m_A e_i = e_i$ for all $i \in A$, $e_j m_A = m_A e_j = 0$ for all $j \in I - A$, and $m_A + m_{I-A} = {}^m I$.

Hence $R/(\sum_{i \in I} e_i R + \text{Ker } \varphi)$ where $\varphi : B \rightarrow \prod e_i R$ such that $\varphi(x) = \langle e_i x \rangle$ $(x \in R)$ is not quasi-injective. It contradicts the fact that R is a right qc-ring. Hence R contains only finitely many nonzero orthogonal idempotents. Hence R is semi-perfect.

By [2], R is noetherian. Hence by proposition 2 R is an SSI-ring. (2) implies (3). Since R is right noetherian and regular. This implies that R is semi-simple artinian. (3)$\Rightarrow$(1) is obvious.

Corollary. R is a V-ring. Then R is right qc if and only if R is left qc.

Proposition 3. Let R be a V-ring and R_n the $n \times n$ matrices over R. Then R is right qc if and only if R_n is right qc.

Proof. Let us suppose that R_n is right qc but R is not right qc. Then there exists a large right ideal A of R such that $A \neq R$. Let (e_{ij}) $(1 \leq i, j \leq n)$ be the matrix units of R_n. Let $I = \{a_{ij} e_{ij} | a_{ij} \in A (1 \leq j \leq n) e_{ij} \in R (2 \leq i \leq n, 1 \leq j \leq n)\}$. Then I is a right ideal in R_n but I is not two-sided because $e_{nn} \in I$ but $e_{1n} e_{nn} = e_{1n}$ and $e_{1n} \notin I$. Now let $0 \neq x$ and $x = \sum_{ij=1}^{n} b_{ij} e_{ij}$. If $b_{ij} = 0$ $(1 \leq j \leq n)$ then $x \in I$, so let $b_{1k} \neq 0$ for some k. Since A is large in R, there is $a \in R$ such that $0 \neq b_{1k} a \in A$. Then $x(ae_{kk})$

$$= \left(\sum_{ij=1}^{n} b_{ij} e_{ij}\right)(ae_{kk}) = \sum_{ij=1}^{n} b_{ik} a e_{kk} \in I.$$

Hence $0 \neq x(ae_{kk}) \in I$. This shows that I is large in R_n but I is not a two-sided ideal of R_n, which contradicts the fact thet every one sided large ideal in R is two-sided. Hence R is right qc. If R is right qc, then R_n is artinian. Hence R_n is right bc.

Proopsition 4. Let R be a duo ring. Then R is right qc if and only if R is right self-injective and every factor ring is injective.

Proof. Let R be right qc. If I is an ideal of R, then factor ring R/I is right qc. Therefore R/I is self-injective.

Conversely suppose that each factor ring of R is injective.

Let M be a cyclic R-module. Then $M \cong R/I$ for some right ideal I of R. By assumption R/I is injective. This implies that R/I is R/I—quasi-injective. Therefore R/I is R quasi-injective. Hence R is a qc-ring.

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