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On the relations of the tensor fields g_{ij} and $^*g_{ij}$ in X_4

By

Choong Hyun Cho

Yonsei University, Seoul, Korea

1. INTRODUCTION

Let X_4 be the space-time endowed with a real nonsymmetric tensor g_{ij} which may be split into its symmetric part h_{ij} and its skew-symmetric part k_{ij}:

(1. 1) a $$g_{ij}=h_{ij}+k_{ij},$$

where

(1. 1) b $$g=\mathrm{Det}((g_{ij}))\neq 0,\quad h=\mathrm{Det}((h_{ij}))\neq 0,\quad k=\mathrm{Det}((k_{ij})).$$

The reciprocal tensor $^*g^{ij}$ of g_{ij}, defined by

(1. 2) $$g_{ij}{}^*g^{ik}=\delta_j^k,$$

may also be decomposed into its symmetric part $^*h^{ij}$ and its skew-symmetric part $^*k^{ij}$:

(1. 3) $$^*g^{ij}={}^*h^{ij}+{}^*k^{ij}$$

Since $\mathrm{Det}((^*h^{ij}))\neq 0$ ([2], p. 41), we may define a unique tensor $^*h_{ij}$ by

(1. 4) $$^*h^{ij}\,{}^*h_{ik}=\delta_k^j.$$

In the present paper, we make an agreement that we use both $^*h^{ij}$ and $^*h_{ij}$, instead of h^{ij} and h_{ij}, as the tensors for raising and/or lowering indices of all tensors defined in X_4 in the usual manner, with the exception of the tensors g_{ij}, h_{ij}, and k_{ij} in order to avoid the notational confusion. We then have for example,

(1. 5) a $$^*k_{ij}={}^*k^{cd}\;{}^*h_{ic}\;{}^*h_{jd},\quad {}^*g_{ij}={}^*g^{cd}\;{}^*h_{ic}\;{}^*h_{jd},$$

so that

(1. 5b) $$^*g_{ij}={}^*h_{ij}+{}^*k_{ij}.$$

V. Hlavaty ([2], p. 8) derived the following relations:

(1. 6) a $$^*h^{ij}=\frac{1}{\bar{g}}\{h^{ij}(1+K)+{}^{(2)}k^{ij}\},$$

(1. 6) b $$^*k^{ij}=\frac{1}{\bar{g}}\left[hk^{ij}+\frac{i}{2}\sqrt{k}\;\;E^{abij}\;k_{ab}\right],$$

where

(1. 6) c $$\bar{g}=g/h,\quad 4K=k_{ij}k^{ij},\quad i=\mathrm{sgn}(E^{abij}\;k_{ab}k_{ij}),$$

and E^{abcd} and e_{abcd} are indicators of density 1 and -1 respectively.

The purpose of the present paper is to derive the remaining relations of the two tensor fields g_{ij} and $^*g^{ij}$ in X_4.

2. RELATIONS OF THE TENSOR FIELDS g_{ij} and $^*g_{ij}$.

In addition to (1.6)a, b, we have the following two theorems:

Theorem (2.1). The tensors $^*h_{ij}$ and $^*k_{ij}$ satisfy the following equations:

(2.1)a $$^*h_{ij}=h_{ij}-{}^{(2)}k_{ij},$$

(2.1)b $$^*k_{ij}=\frac{1}{\bar{g}}\left\{(1-\bar{k})k_{ij}-2(1+K)\,{}^{(3)}k_{ij}+\frac{i}{2}\sqrt{\bar{k}}\left(e_{ijab}k^{ab}+2e_{abc\,[i}{}^{(2)}k^{a}{}_{j]}k^{bc}+e_{abcd}k^{ab\,(2)}k_i{}^{c\,(2)}k_j{}^{d}\right)\right\}.$$

Proof. In order to prove (2.1)a, consider a tensor $^*X_{ij}$ defined by

(2.2)a $$^*h^{ak}\,{}^*X_{aj}=\delta^k_j.$$

Substitution for $^*h^{ak}$ from (1.6)a into (2.2)a gives

(2.2)b $$(1+2K)\,{}^*X_{jk}+{}^{(2)}k_j{}^{a}\,{}^*X_{ka}=\bar{g}\,h_{jk}.$$

Multiplying (2.2)b by ${}^{(2)}k_b{}^j$ first and then substituting for ${}^{(4)}k^{ca}$ from

(2.3) $${}^{(4)}k_i{}^j+2K\,{}^{(2)}k_i{}^j+\bar{k}\,\delta^j_i=0,\quad (\bar{k}=k/h)$$

obtained in [2], p. 23, we have

(2.2)c $$-\bar{k}\,{}^*X_{ij}+{}^{(2)}k_i{}^{a}\,{}^*X_{aj}=\bar{g}\,{}^{(2)}k_{ij}.$$

(2.1)a is easily obtained by solving (2.2)b, c for $^*X$ and observing (1.4) and (2.2)a. In order to prove (2.1)b, we have by using (1.6)b and (2.1)a

(2.4) $$^*k_{ij}={}^*h_{ia}\,{}^*h_{jb}\,{}^*k^{ab}=(h_{ia}-{}^{(2)}k_{ia})(h_{jb}-{}^{(2)}k_{jb})\frac{1}{g}\left(hk^{ab}+\frac{i}{2}\sqrt{\bar{k}}\,E^{abcd}k_{cd}\right).$$

This equation leads at once to (2.1)b if we make use of (2.3).

Theorem (2.2). We have

(2.5)a $$h_{ij}=\frac{1}{^*g}\{^*h_{ij}(1+2\,{}^*K)+{}^{(2)*}k_{ij}\},$$

(2.5)b $$h^{ij}={}^*h^{ij}-{}^{(2)*}k^{ij},$$

(2.5)c $$k_{ij}=\frac{1}{^*g}\left({}^*h\,{}^*k_{ij}+\frac{^*i}{2}\sqrt{^*k}\;{}^*e_{abij}\,{}^*k^{ab}\right),$$

(2.5)d $$k^{ij}={}^*\frac{1}{\bar{g}}\left\{(1-{}^*\bar{k})\,{}^*k^{ij}-2(1+{}^*K)\,{}^{(3)*}k^{ij}+\frac{^*i}{2}\sqrt{^*\bar{k}}\left({}^*E^{ijab}\,{}^*k_{ab}+2\,{}^*E^{abc\,[i}\,{}^*k_a{}^{j]}\,{}^*k_{bc}+\right.\right.$$
$$\left.\left.+{}^*E^{abcd}\,{}^*k_{ab}\,{}^{(2)*}k_c{}^{i\,(2)*}k_d{}^{j}\right)\right\}.$$

Proof. The proof is similar to that of (1.6) and (2.1).

REFERENCES

1. L.P. Eisenhart, Riemannian Geometry, Princeton Unversity Press, 1949.
2. V. Hlavaty, Geometry of Einstein's Unified Field Theory, P. Noordhoff Ltd., 1957.