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On the property of closure and interior operators

by

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1. Introduction.

If E is the set of rational number in the space of reals R, then $CiE=\phi$ $iCE=R$ where C denotes the closure operator and i denotes the interior operator. It is evident that the closure and interior operators does not commutes.

In this note, we will give a characterization of property of commutativity of closure and interior operators in a general topological space.

2. Definitions and lemmas.

Definition A set E is a subset of topological space T. E has property X if and only if $CiE=iCE$.

Lemma 1. For E a subset of a topological space T. E has property X if and only if kE has property X, where k is the complement operator.

Proof. Let E has property X, then $CkE=kCkCkE=kiCkkE=CikE$.

Let kE has property X. Then $E=kkE$, therefore E has property X.

Lemma 2. Let E be a subset of a space T, then E has property X implies that CE has property X.

Proof. $CiCE=CiCkkE=CkCikE=CkiCkE=CkkCkCkE$ difference operator. Then

(1) $E\triangle F=F\triangle E$

(2) $(E\triangle F)\triangle G=E\triangle(F\triangle G)$ and $E\cap(F\triangle G)=(E\cap F)\triangle(E\cap G)$

3. Theorems

We will give a characterization of property X in a topological space T.

Theorem 1. Let E be a subset of a topological space R. Then E has property X if and only if $E=(A-B)\cup(B-A)$ where A is both open and closed and B is nowhere dense.

Proof. Sufficiency. Let $E=(A-B)\cup)B-A)$ where A is open and closed and B is nowhere dense. Now $A-B\subset A$ and $B-ACkA$ Hence

$$CiE=Ci\{(A-B)\cup(B-A)\}=C\{i(A-B)\cup i(B-A)\}$$
$$=C\{i_A(A-B)\cup ik_A(B-A)\}=Ci_A(A-B)\cup Cik_A(B-A)$$
$$=C_Ai_A(A-B)\cup Ck_Aik_A(B-A)$$

Similarly $iCE = i_A C_A(A-B) \cup ik_A Ck_A(B-A)$. Now $A-B = A-A\cap B$ and $B\cap A$ is nowhere dense in A.

Then $A-B$ is the complement relative to A of a set nowhere dense in A and $C_A i_A(A-B) = i_A C_A(A-B)$. Moreover $B-A = B\cap kA$ is nowhere dense in kA.

Thus $Ck_A ik_A(B-A) = ik_A Ck_A(B-A)$ therefore we obtain $CiE = iCE$. $CiE = iCE = iCCE$.

The converse of Lemma 2 is false. For E is the set of rationals in the reals R, then E not have property X, but $CE = R$ and R has property X.

Lemma 3. Let E be a subset of space T.

(1) E has property X implies that iE has property X.

(2) E or kE are nowhere dense then E has property X.

Proof. (1) Assume E has property X, then $iE = kCkE$ has property X.

(2) Let E be a nowhere dense in T, then $CiE = \phi$. But $iE \subset iCE = \phi$ and $CiE = \phi$ therefore E has property X.

Remark. The converse of (1) is false. For let E be the set of rationals in the real R. Then E does not have property X, but $iE = \phi$ has property X.

Lemma 4. (1) Let A be a closed in T and $B\subset A$. Then $CB = C_A B$ where C_A is the closure operator in the subspace A.

(2) Let A be open in T, and $B\subset A$, Then $iB = i_A B$ where i_A denotes the interior operator in the subspace A.

Lemma 5. Let A be open and closed in a space T.

(1) Let $B\subset A$ and $C\subset kA$, Then $i(B\cup C) = iB\cup iC$.

(2) B be a nowhere dense in T, then $B\subset A$ is nowhere dense in A.

Lemma 6. Let E, F and G are any sets. Let $\triangle$ be the symmetric. Necessity, Suppose E has property X, and let $A = CiE = iCE$ Then A is both open and closed in R.

We now show that $E-A$ and $A-E$ areeach nowhere dense, $iC(E-A) \subset iCE = A$, but also $iC(E-A) \subset iC(kA) = kA$.

Hence $iC(E-A) = \phi$. In addition, $iC(A-E) \subset iCA = A$ and also $iC(A-E) \subset iCkE = kCkCkE = kCiE = kA$. Hence $iC(A-E) = \phi$. Let $B = (A-E)\cup(E-A)$.

Then B is nowhere dense and the proof will be complete when we show that $E = (A-B)\cup(B-A)$. To this end we observe that

$$
\begin{aligned}
(A-B)\cup(B-A) &= (A\cap kB)\cup(B\cap kA)\\
&= [A\cap k\{(A\cap kE)\cup(E\cap kA)\}]\cup[\{(A\cap kE)\cup(E\cup kA)\}\cap kA]\\
&= [A\cap k(A\cap kE)\cap k(E\cap kA)]\cup[E\cap kA]\\
&= [A\cap(kA\cup E)\cap(kE\cup A)]\cup[E\cap kA]\\
&= [A\cap(kA\cup E)\cap(kE\cup A)]\cup[E\cap kA]\\
&= [(A\cap E)\cap(kE\cup A)]\cup[E\cap kA]\\
&= (A\cap E)\cup(E\cap kA) = E
\end{aligned}
$$

Theorem 2. Let E and E′ has property X in a space T. Then $E\cap E'$ has property X.

Proof. $E = A\triangle B$, and $E' = A'\triangle B'$ where A and A′ are both open and closed and B and B′ are both nowhere dense by theorem 1.

Then $E \cap E' = E \cap (A' \triangle B')$

$$= (E \cap A') \triangle (A' \triangle B') = (E \cap A') \triangle (E \cap B')$$

$$= \{(A \cap A') \triangle (B \cap A')\} \triangle (A \cap B') \triangle (B \cap B')\}$$

$$= (A \cap A') \triangle \{(B \cap A') \triangle (A \cap B') \triangle (B \cap B')\} = A'' \triangle B''$$

Whence $A'' = A \cap A'$ and $B'' = (B \cap A') \triangle (A \cap B') \triangle (B \cap B')$.

But A'' is both open and closed and B'' is nowhere dense. $E \cap E'$ has property X by Theorem F.

REFERENCES

1. J.L. Kelley, General Topology, New York 1955.
2. J. Dugundji, Topology, Boston 1966.
3. W. Sierpinski, General Topology, Toronto 1952.