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Nonlinear Semigroup and Dissipative Operators.

by

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Introduction

This paper is concerned with the behavior of semigroups of nonlinear contraction operator from a subset of a Banach space into itself. Recently, many results, known for semigroup of linear operator, were extended to nonlinear semigroups. The representations of nonlinear semigroup were given by I. miyadera and S. oharu [1] and I. miyadera [2].

We start §1. with the notion of a multi-valued nonlinear dissipative. §2. is concerned with resolvent of dissipative. §3. is devoted to generation of nonlinear semigroup on a subset of a Banach space with its dual uniformly convex and a general Banach space.

Dissipative operators.

Let X be a Banach space with its dual X^*. We denote by (x, f) the value of $f\in X^*$ at $x\in X$. The norms in X and X^* are denote by $\|\ \|$. We find it convenient to use the notation

$$|||X_0||| = \inf\{\|x\|; x\in X_0\}$$

for any nonempty subset X_0 of X.

Definition 1.1. The duality map F of X is the multi-valued mapping from X into X^* defined by

$$Fx = \{f\in X^*; \mathrm{Re}(x, f) = \|x\|^2 = \|f\|^2\}$$

Definition 1.2. An operotor A in X is said to be dissipative if every $x, y\in D(A)$, $x'\in Ax$ and $y'\in Ay$, there exists $f\in F(x-y)$ such that

$$\mathrm{Re}(x'-y', f)\leqq 0.$$

We say A is m-dissipative if A is dissipative and $R(I-\alpha A) = X$ for all $\alpha>0$.

Theorem 1.1. Let $x, y\in X$. Then $\|x\|\leqq\|x+\alpha y\|$ for every $\alpha>0$ if and anly if there is $f\in Fx$ such that $\mathrm{Re}(y, f)\geqq 0$.

Proof. The assertion is trivial if $x=0$, so we shall assume $x\neq 0$ in the following. If $\mathrm{Re}(y, f)\geqq 0$ for some $f\in Fx$, then

$$\|x\|^2 = (x, f) = \mathrm{Re}(x, f) \leqq \mathrm{Re}(x+\alpha y, f) \leqq \|x+\alpha y\|\|f\| \text{ for } \alpha>0.$$

Since $\|f\| = \|x\|$, we obtain $\|x\|\leqq\|x+\alpha y\|$

Suppose, conversely, that $\|x\|\leqq\|x+\alpha y\|$ for $\alpha>0$. For each$\alpha>0$. Let $f\in F(x+\alpha y)$ and $g = f_\alpha/\|f_\alpha\|$ so that $\|g_\alpha\| = 1$. Then $\|x\|\leqq\|x+\alpha y\| = (x+\alpha y, g_\alpha) = \mathrm{Re}(x, g_\alpha) + \alpha\mathrm{Re}(y, g_\alpha) \leqq \|x\| + \alpha\mathrm{Re}(y, g_\alpha)$.

Thus

(1.1) $$\liminf_{\alpha\to 0} \mathrm{Re}(x, g_\alpha) \geqq ||x|| \text{ and } \mathrm{Re}(y, g_\alpha) \geqq 0.$$

Since the closed unit fall of X^* is compact in the weak topology, the net $\{g_\alpha\}$ has a cluster point $g \in X^*$ with $||g|| \leqq |$. In view of (1.1), however, g satisfies

$\mathrm{Re}(x, g) \geqq ||x||$ and $\mathrm{Re}(y, g) \geqq 0$. Hence we must have $||g||=1$ and $(x, g)=||x||$.
on setting $f=||x||g$, we see that $f \in Fx$ and $\mathrm{Re}(y, f) \geqq 0$.

Theorem 1.2. If X^* is uniformly convex, F is single-valued and is uniformly continuous on any bounded set of X. In other words, for each $\varepsilon>0$ and $M>0$, there is $\delta>0$ such that $||x||<M$ and $||x-y||<\delta$ implies $||Fx-Fy||<\varepsilon$.

Proof. It suffices to show that the assumptions

$$||x||<M,\ ||x_n-y_n||\to 0,\ ||Fx_n-Fy_n||\geqq \varepsilon_0>0,\ n=1, 2, \cdots\cdots$$

lead to a contradiction. If $x_n \to 0$, then $y_n \to 0$ and so $||Fx_n||=||x_n||\to 0$ and similarly $||Fy_n||\to 0$, hence $||Fx_n-Fy_n||\to 0$, a contradiction.

Thus we may assume that $||x_n||\geqq \alpha>0$, replacing the given sequence by suitable subsequence if necessary. Then $||y_n||\geqq \frac{\alpha}{2}$ for sufficiently large n. Set

$u_n=\frac{x_n}{||x_n||}$ and $v_n=\frac{y_n}{||y_n||}$. Then $||u_n||=||v_n||=1$ and $u_n-v_n=(x_n-y_n)/||x_n||$
$+(||x_n||^{-1}-||y_n||^{-1})y_n$ so that $||u_n-v_n||\leqq 2||x_n-y_n||/||x_n|\to 0$.

since $Fx_n=F(||x_n||u_n)=||x_n||Fu_n$ and similarly $Fy_n=||y_n||Fv_n$, we now obtain

$$Fx_n-Fy_n=||x_n||(Fu_n-Fv_n)+(||x_n||-||y_n||)Fv_n \longrightarrow 0 \text{ by } ||x_n||<M.$$

Thus we have arrived at a contradiction again.

Theoren 1.3. Let $x(t)$ be an X valued function on an interval of real numbers.
Suppose $x(t)$ has a week derivative $x'(s)\in X$ at $x=s$.
If $||x(t)||$ is also differentiable at $t=s$, then

(2.1) $$x(s)\left(\frac{d}{ds}\right)||x(s)||=\mathrm{Re}(x'(s), f)$$

for every $f\in Fx(s)$.

Proof. Since $\mathrm{Re}(x(t), f)\leqq ||x(t)||\ ||f||=||x(t)||\ ||x(s)||$ and $\mathrm{Re}(x(s), f)=||x(s)||^2$, we have

$$\mathrm{Re}(x(t)-x(s), f)\leqq ||x(s)||(||x(t)||-||x(s)||).$$

Dividing both side by $t-s$ and letting $t-s$ from above and from below, we obtain

$$\mathrm{Re}(x'(s), f)\leqq ||x(s)||\left(\frac{d}{ds}\right)||x(s)||$$

2. Resolvents of dissipative

Definition 2.1. Let X_0 be a subset of X. Let $\mathrm{Cont}(X_0)$ be the set of all contraction operators from X_0 into with domain X_0 if and only if $||Tx-Ty||\leqq ||x-y||$ for $x, y\in X_0$.

Let A be dissipative. Then we can define

$$J_\alpha=(I-\alpha A)^{-1}$$

for all $\alpha>0$, with $D(J_\alpha)=R(I-\alpha A)$ and $R(J_\alpha)=D(a)$, and set $A_\alpha=\alpha^{-1}(J_\alpha-I)$. Both J_α and A_α

are single-valued with $D_\alpha = D(J_\alpha) = D(A_\alpha)$. By definition $J_\alpha x = y$ for $x \in D_\alpha$ if and only if $x \in (I - \alpha A) y = y - \alpha A y$ for $y \in D(A)$. J_α is contraction and A_α is Lipschiz continuous;

$$||A_\alpha x - A_\alpha y|| \leqq 2\alpha^{-1} ||x - y||$$

for $x, y \in D$.

Theorem 2.1. Let A be dissipative and let $\alpha > 0$. Then

i) A is dissipative

ii) $A_\alpha x \in AJ_\alpha$ and $|||AJ_\alpha x||| \leqq ||A_\alpha x||$ for $x \in D_\alpha$,

iii) if $x \in D(A) \cap D_\alpha$, then

$$|||AJ_\alpha x||| \leqq ||A_\alpha x|| \leqq |||Ax|||.$$

Proof. (i) Let $x, y \in D_\alpha$ and $f \in F(x-y)$. Then

$$\begin{aligned}
&\mathrm{Re}(A_\alpha x - A_\alpha y, f) \\
&= \alpha^{-1}\mathrm{Re}(J_\alpha x - J_\alpha y, f) - \alpha^{-1}\mathrm{Re}(x - y, f) \\
&\leqq \alpha^{-1} ||J_\alpha x - J_\alpha y|| \; ||f|| - \alpha^{-1} ||x - y||^2 \\
&= 0.
\end{aligned}$$

(ii) Set $J_\alpha x = y$. Then $x = y - \alpha y'$, with $y' \in Ay$.
Hence $A_\alpha x = \alpha^{-1}(y - x) = y'$, $Ay = AJ_\alpha x$, and

$$|||AJ_\alpha x||| \leqq ||y'|| = ||A_\alpha x||.$$

(iii) if $x \in D(A) \cap D_\alpha$, then $x = J_\alpha(x - \alpha x')$ for any $x' \in Ax$. we have

$$\begin{aligned}
&||A_\alpha x|| = \alpha^{-1} ||J_\alpha x - x|| \\
&= \alpha^{-1} ||J_\alpha x - J_\alpha(x - \alpha x')|| \\
&\leqq ||x'||
\end{aligned}$$

Hence $||A_\alpha x|| \leqq \inf_{x' \in Ax} ||x'|| = ||Ax||$.

Theorem 2.2. Let A be dissipative. Then

(i) $\lim J_\alpha x = x$ for $x \in D(A) \cap D_\alpha$ and $\alpha > 0$,

(ii) if n is a positive integer, for $x \in D(A) \cap D(J_\alpha^n)$ and $\alpha > 0$,

Proof. If follows from $||J_\alpha x - x|| \leqq \alpha |||Ax|||$

3. Generation of semigroups.

Let X_0 be a subset of X. Let $\{T(t); t \leqq 0\}$ be a family of one-parameter operators, not necessarily linear, from X_0 into itself satisfying the following conditions

i) $T(0) = I$, $T(s)t(t) = T(t+s)$ for $s, t \geqq 0$,

ii) for $x \in X_0$, $T(t)x$ is strongly continuous in $t \geqq 0$,

iii) $T(t) \in \mathrm{Cont}(X_0)$ for $t \geqq 0$

Then we call such a family $\{T(t); \geqq 0\}$ a nonlinear contraction semigraup on X_0. And we define infinitesimal generator of the semigroup $\{T(t); t \geqq 0\}$ on X_0 by

$$A_0 x = \lim_{h \to 0+} \frac{T(h) - I}{h} x$$

and the weak infinitesimal generator A' by

$$A'x = w\text{-}\lim_{h\to 0+} \frac{T(h)-I}{h}x, \text{ if the right sides exists in } X.$$

Theorem 3.1. A and A′ are dissipative.

Proof. We set $A^h = \frac{T(h)-I}{h}$ for $h>0$.

For $x, y \in D(A)$, we have $\mathrm{Re}(A^h x - A^h y, f) \leqq 0$ for every $f \in F(x-y)$, and hence $\mathrm{Re}(A'x - A'y, f) \leqq 0$ for every $f \in F(x-y)$ as $h \to 0+$. Thus A′ is dissipative. Since $A' \supset A_0$, A_0 is dissipative.

Theorem 3.2. Let A be m-disspative and let $\alpha > 0$. Then equation

$$\frac{du(t;x)}{dt} = Au(t;x) \text{ for } t>0 \tag{3.1}$$

$$u(0;x) = x \in X$$

has a unique solution $u(t;x) \in C'([0,\infty);x)$ for $x \in X$, and

$$||u(t;x) - u(t;y)|| \leqq ||x-y||$$

for $x, y \in X$.

Proof. Since A_α x is Lipschitz continuous, uniformly in X, the equation (3.1) has a unique solution $u(t;x) \in C'([0,\infty);X)$ for $x \in X$. For $x, y \in X$, put $w(t) = u(t;x) - u(t;y)$ for $t \geqq 0$. Then $w(t) \in C'([0,\infty);X)$ and

$$\frac{d}{dt}w(t) = A_\alpha u(t;x) - A_\alpha u(t;y)$$

$$w(0) = x - y.$$

Since $||w(t)||$ is absolutely continuous, $||w(t)||$ is differentiable for a.e. $t>0$. Tuse by Theorem 1.3. we get for a.e. $s>0$.

$$||w(s)|| \frac{d}{ds}||w(s)|| = \mathrm{Re}(w'(s), f) \tag{3.2}$$

$$= \mathrm{Re}(A_\alpha u(s;x) - A_\alpha u(s;y), f)$$

for every $f \in Fw(s)$. Since A_α is dissipative by Theorem 2.1 (i) there exists $f_s \in Fw(s)$ such that

$$\mathrm{Re}(A_\alpha u(s;x) - A_\alpha u(s;x) - A_\alpha u(s;y), f_s) \leqq 0.$$

Thus

$$||w(s)|| \frac{d}{ds}||w(s)|| \leqq 0$$

for a.e. $s \geqq 0$ combining this with (3.2). Since

$$||w(t)||^2 - ||w(0)||^2 = \int_0^t \frac{d}{ds}||w(s)||^2 ds \leqq 0$$

for $t \geqq 0$, we abtain

$$||u(t;x) - u(t;Y)|| \leqq ||x-y||$$

for $t \geqq 0$.

Theorem 3.3. Let A be m-disspative and $\alpha > 0$. Then there exists a semigroup $\{T_\alpha(t); t \geqq 0\}$ on X with its infinitesimal generator A_α such that for $x \in X$, $T_\alpha(t)x \in C'([0,\infty);X)$ and

$$\frac{d}{dt}T_\alpha(t)x = A_\alpha T(t)x$$

for $t \geqq 0$.

Proof. Let $u(t;x)$ be the uniqe solution of the equation (3. 1), and put $T_\alpha(t)x=u(t;x)$ for $x\in X$ and $t\geqq 0$. Then $\{T_\alpha(t);t\geqq 0\}$ is the desired semigroup on X.

Theorem 3. 4. Let X^* uniformly convex and let A is m-dissipative. Then there exists a semigroup $\{T(t);t\geqq 0\}$ on $D(a)$.

Proof. By theorem 3. 3, we note that

(3. 3) $$||A_\alpha T_\alpha(t)x||\leqq||A_\alpha x||\leqq|||Ax|||$$

for $x\in D(A)$ and $t\geqq 0$. For $x\in D(A)$, put

$$w_{\alpha,\beta}(t)=T_\alpha(t)x-T_\beta(t)x$$

for $\alpha,\beta>0$, in Theorem 3. 3.

We shall show that $\lim_{\alpha,\beta\to 0+} w_{\alpha\beta}(t)=0$ for $t\geqq 0$. By (3. 3), we have

$$||w_{\alpha,\beta}(s)||\leqq\int_0^t||A_\alpha T_\alpha(t)x-A_\beta T_\beta(t)x||dt$$

$$\leqq 2||Ax||s$$

for $s\geqq 0$. Setting $v_{\alpha,\beta}(s)=J_\alpha T_\alpha(s)x-J_\beta T_\beta(s)x$, we obtain

$$||w_{\alpha,\beta}(s)-v_{\alpha,\beta}||\leqq(\alpha+\beta)\,|||Ax|||,$$

and hence we have

$$||v_{\alpha,\beta}(s)||\leqq(2s+\alpha+\beta)\,|||Ax|||.$$

By the dissipative of A, combining with Theorem 2. 1 (ii),

(3. 4) $$\mathrm{Re}(A_\alpha T_\alpha(s)x-A_\beta T_\beta(s)x,\ Fv_{\alpha,\beta}(s))\leqq 0.$$

Thus $\mathrm{Re}(A_\alpha T_\alpha(s)x-A_\beta T_\beta(s)x,\ FW_{\alpha,\beta}(s))$

$$\leqq\mathrm{Re}(A_\alpha T_\alpha(s)x-A_\beta T_\beta(s)x,\ Fw_{\alpha,\beta}(s)$$

$$\leqq 2\,|||Ax|||\ ||Fw_{\alpha,\beta}(s)-Fv_{\alpha,\beta}(s)||,$$

here we have used (3. 3) and (3. 4) Since $||w_{\alpha,\beta}(s)||$ is absolutely continuous in $s\geqq 0$, $||w_{\alpha,\beta}(s)||$ is differentiable at a. e. $s\geqq 0$. Hence

$$||w_{\alpha,\beta}(t)||^2=\int_0^t\frac{d}{ds}||w_{\alpha,\beta}(s)||^2ds$$

$$=2\int_0^t\mathrm{Re}(A_\alpha T_\alpha(s)x-A_\beta T_\beta(s)x,\ Fw_{\alpha,\beta}(s))ds$$

$$\leqq 4\,|||Ax|||\int_0^t||Fw_{\alpha,\beta}(s)-Fv_{\alpha,\beta}(s)||ds.$$

Fix $t>0$. It follow that the set

$$\{w_{\alpha,\beta}(s);v_{\alpha,\beta}(s);0\leqq s\leqq t,\ 0<\alpha,\beta\leqq 1\}$$

is bounded and $w_{\alpha,\beta}(s)-v_{\alpha,\beta}(s)$ converges uniformly in $s\in[0,t_0]$ to zero as $\alpha,\beta\longrightarrow 0+$. By Theorem 1. 2 since F is uniformly continuous on any bounded set, we obtain that

$$\lim_{\alpha,\beta\to 0} w_{\alpha,\beta}(t)=0$$

uniformly in $t\in[0,t_0]$, that is, for $x\in D(A)$

(3. 5) $$\lim_{\alpha,\beta\to 0}||T_\alpha(t)x-T_\beta(t)||x=0$$

uniformly in $t\in[0,t_0]$. Since $T_\alpha(t)\in\mathrm{Cont}(X)$. (3. 5) holds true for $x\in D(A)$. we define $T(t)$ by

$$T(t)x=\lim_{\alpha\to 0}T_\alpha(t)x$$

for $x\in D(A)$. Then $\{T(t);t\geqq 0\}$ is semigroup on $\overline{D(A)}$.

Indeed, it is sufficient to show that $T(t)$ maps $\overline{D(A)}$ into itself.
For $x \in D(A)$, since $||J_\alpha T_\alpha(t)x - T_\alpha(t)x|| \leqq \alpha |||Ax|||$, we have

$$\lim_{\alpha \to 0} J_\alpha T_\alpha(t)x = T(t)x$$

uniformly in $t \in [0, t_0]$, and $T(t)x \in \overline{D(A)}$
because $J_\alpha T_\alpha(t) \in D(A)$. Since $T(t) \in \text{Cont}\ (D(A))$, $T(t)$ maps $\overline{D(A)}$ into $\overline{D(A)}$.

Reference

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要　約

本論文에서는 Banach 空間에서의 非線形 縮小作用素의 半群에 대하여 조사하고 非線形 半群에서의 生成作用素의 生成을 論하였다.