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# A note on locally compactness of cluster set

by

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Let $(X, T)$ be a topological space. We denote by $X^*$ the set of all filters on X which converge for T to some point of X. We regard convergence of filter for T as a relation $f: X^* \to X$ writing $\varphi f x$ if $\varphi$ converges to x, where $\varphi$ is member of $X^*$

This relation is a mapping if and only if $(X, T)$ is a Hausdorff space, Wyler [1] who has shown that a Hausdorff space $(X, T)$ is regular if and only if convergence of filter on X for T defines a continuous map $f: (X^*, T^*) \longrightarrow (X, T)$ and also generalize above paragraph.

That is, for a topological space $(X, T)$ with filter convergence f, $(X, T)$ is a $T_3$-space if and only if $f: (X^*, T^*) \to (X, T)$ is continuous.

**Definition 1.** For a filter $\mathfrak{F}$ in a topological space X the cluster set of $\mathfrak{F}$ is $\bigcap \{\bar{F} : F \in \mathfrak{F}\}$. We will denote the cluster set of $\mathfrak{F}$ by $\beta(\mathfrak{F})$.

For a set X, and subset U of X, $P(x)$ is the power set of X and $I(U, X)$ is the family of all non-empty subsets of X which intersect U.

**Definition 2.** X is topological space, the lower semifinite topology on $P(X)$ has a subbasis all sets of the form $I(U, X)$ where U is open in X.

We will denote all filters of a topological space X which have non-empty cluster set by $X^\#$. The filter space $X^\#$ will be assumed to carry the topology which has as a subbasis all sets of the $U^\# = \{\mathfrak{F} \in X^\# : F \cap U \neq \phi$ for all F in $\mathfrak{F}\}$ where U is open in X.

**Theorem 3.** Let X be a Hausdorff space and $P(X)$ have the semifinite topology. Then the cluster set function $f: X^\# \to P(X)$ is continuous if and only if X is locally compact.

**Proof.** Assume X is not locally compact we will prove that f is not continuous. Since X is not locally compact, there is a point p in X such that no nbd of p is compact. Hence for every nbd of p, there is a filter $\mathfrak{F}_U$ in U which has no cluster point.

Let $\mathfrak{F}_0$ be the filter of all supersets of $\{p\}$. If $W^\#$, where W is open, is a subbasis nbd of $\mathfrak{F}_0$, then in particular $\{p\} \cap W \neq \phi$ so W is a nbd of p. Note since X is Hausdorff, $f(\mathfrak{F}_0) = \{p\}$.

Now let $I(V, X)$, where V is open, be a subbasic nbd of $f(\mathfrak{F}_0)$, so that V is a nbd of p. Consider for each nbd U of p, the filter $Z_U = \{F \cup (X-V) : F \in \mathfrak{F}_U\}$ which has cluster set $f(Z_U) = X - V$. If $W^\#$ is a subbasic nbd of $\mathfrak{F}_0$, $Z_U \in W^\#$ for $U \subset W$.

Thus the net of filter $Z_U$ converge to $\mathfrak{F}_0$ But $f(Z_U)$ does not belong to $I(V, X)$ for any U. Therefore $f : X^{\#} \rightarrow P(X)$ is not continuous.

Assume X is locally compact. Let $\mathfrak{F}_0$ be in $X^{\#}$ and $I(V, X)$, where V is open, be a subbasic nbd of $f(\mathfrak{F}_0)$.

Then for some p in $f(\mathfrak{F}_0)$, p is also in V. Hence there is compact nbd of U of p contained in V.

Consider the nbd of $\mathfrak{F}_0$, $U^{\#}$, and let $\mathfrak{F}$ be in this nbd. Then for each F in $\mathfrak{F}$, $F \cap U \neq \phi$ so that a filter $\mathfrak{F}_U$ is generated by the collection $\{F \cap U : F \in \mathfrak{F}\}$ Since U is compact, $\mathfrak{F}_U$ must have a cluster point q in U.

But $\mathfrak{F}$ is coarser than $\mathfrak{F}_U$, so q is also a cluster point of $\mathfrak{F}$. Thus $f(\mathfrak{F}) \cap U \neq \phi$ so $f(\mathfrak{F}) \in I(V, X)$. Therefore f is continuous.


## 要 約

Wyler [1]는 Hausdorff space의 regularity를 論함에 있어서 Filter공간상에서 相異한 位相으로 취급하였으나 본 論文에서는 同一한 位相을 使用하여 취급하였다.


## REFERENCES


1. Oswald Wyler, A characterization of regularity in topology. Proc. Amer Math Soc. 29(1971).
2. N. Bourbaki: Elements of Mathematics I.