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Some fixed point theorems in a metric space.

By

Kwon, Young-In

Kyungpook University, Taggu, Korea

1. Introduction.

Let T be a self-mapping on a non-empty complete metric space (X, d).

Let a_i, $i=1, 2, \cdots, 5$ be non-negative real numbers such that $\sum_{i=1}^{5} a_i < 1$ and for any distinct x, y in X,

(1) $d(T(x), T(y)) \leqq a_1 d(x, y) + a_2 d(x, T(y)) + a_3 d(y, T(x)) + a_4 d(x, T(x)) + a_5 d(y, T(y))$, where $a_i = \alpha_i(x, y)$.

It is the purpose of this paper to obtain some fixed point theorems for a self-mapping T in (1) with a_i replaced by $\alpha_i(d(x, y))/d(x, y)$.

It is introduced [4] self-mappings α_i, $i=1, \cdots, 5$, on $[0, \infty)$ such that $\alpha_2=\alpha_3, \alpha_5=\alpha_5, \sum_{i=1}^{5}\alpha_i(tt) <$ for all $t>0$ and each α_i is upper semicontinuous.

It is assumed that for any distinct x, y in X, (1) is satisfied with a_i replaced by $\alpha_i(d(x, y))/d(x, y)$. We proved that T has a unique fixed point for each α_i is lower semicontinuous from right.

2. **Theorem.** Let T be a generalized nonexpansive mapping on a complete metric space. Suppose $\alpha_2=\alpha_3, \alpha_4=\alpha_5$. Then T has a unique fixed point.

Proof. Let $x_0 \in X$. Define $x_{2n+1}=T(x_{2n})$, $x_{2n+2}=T(x_{2n+1})$, $n=0, 1, 2, \cdots$. We may assume that $x_n \neq x_{n+1}$ for each n. From that T is a generalized nonexpansive mapping with $a_i=\alpha_i(x_0, x_1)$.
$d(x_1, x_2) = d(T(x_0), T(x_1)) \leqq (a_1+a_4) d(x_0, x_1) + a_2 d(x_0, x_2) + a_5 d(x_1, x_2)$.
Since $d(x_0, x_2) \leqq d(x_0, x_1) + d(x_1, x_2)$,

(1) $d(x_1, x_2) \leqq \frac{a_1+a_2+a_4}{1-a_2-a_5} d(x_0, x_1)$. From the hypothesis, $a_2+a_4 \leqq r/2 < 1/2$ and

(2) $\frac{a_1+a_2+a_4}{1-a_2-a_5} \leqq \frac{r-a_2-a_4}{1-a_2-a_4} \leqq \max\left\{\frac{r-x}{1-x} : x \in [0, 1/2]\right\} \leqq r.$

From (1) and (2), $d(x_1, x_2) \leqq r d(x_0, x_1)$. By induction,

(3) $d(x_{n+2}, x_{n+1}) \leqq r d(x_{n+1}, x_n)$ $n=0, 1, 2, \cdots$ and $d(x_{n+1}, x_n) \leqq r^n d(x_0, x_1)$ $n=0, 1, 2, \cdots$.

Since $r<1$, $\sum_{n=0}^{\infty} d(x_{n+1}, x_n) < \infty$ and therefore $\{x_n\}$ is Cauchy.

By completeness of (X, d), $\{x_n\}$ converges to some point x in X. Since $x_{n+1} \neq x_n$ for each n, we may assume that $x_{2n+1} \neq x$. Thus there is a subsequence $\{k(n)\}$ of $\{n\}$ such that $x_{2k(n)+1} \neq x$ for each n. Let $n \geqq 1$,

(4) $d(x, T(x)) \leqq d(x, x_{2k(n)+1}) + d(x_{2k(n)+1}, T(x)) = d(x, x_{2k(n)+1}) + d(T(x_{2k(n)}), T(x))$.

From the hypothesis with $a_i = \alpha_i(x_{2k(n)}, x)$,

(5) $d(T(x_{2k(n)}), T(x)) \leqq a_1 d(x_{2k(n)}, x) + a_2 d(x_{2k(n)}, T(x)) + a_3 d(x, x_{2k(n)+1}) + a_4 d(x_{2k(n)}, x_{2k(n)+1})$
$+ a_5 d(x, T(x)) \leqq d(x_{2k(n)}, x) + r/2 d(x_{2k(n)}, T(x)) + d(x, x_{2k(n)+1}) + d(x_{2k(n)}, x_{2k(n)+1}) + r/2 d(x, T(x))$.

By (4), (5) and letting $n \to \infty$, $d(x, T(x)) \leqq r d(x, T(x))$. Since $r < 1$, $T(x) = x$. If $T(y) = y$, $d(x, y) = d(T(x), T(y)) \leqq (a_1 + a_2 + a_3) d(x, y) < d(x, y)$, a contradiction.

Let X be a complete metric space. Let T be a self-mapping on X. T is called a generalized non-expansive mapping if there exists symmetric functions α_i, $i = 1, 2, \cdots 5$ of $X \times X$ into $[0, \infty]$ such that

(a) $r \equiv \sup \left\{ \sum_{i=1}^{5} \alpha_i(x, y); x, y \in X \right\} < 1$. and

(b) for any distinct x, y in X,
$d(T(x), T(y)) \leqq a_1 d(x, y) + a_2 d(x, T(y)) + a_3 d(y, T(x) + a_4 d(x, T(x)) + a_5 d(y, T(y))$, where $a_i = \alpha_i(x, y)$.

Theorem. Let T be a self-mapping on a complete metric space (X, d). Suppose that lower semi-continuous from the right functions α_i, $i = 1, 2, \cdots 5$ of $(0, \infty)$ into $[0, \infty]$ such that

(1) $\sum_{i=1}^{5} \alpha_i(t) < t, t > 0$;

(2) for any distinct x, y in X,
$d(T(x), T(y)) \leqq a_1 d(x, y) + a_2 d(x, T(y)) + a_3 d(y, T(x)) + a_4 d(x, T(x)) + a_5 d(y, T(y))$, where $a_i = \alpha_i(x, y) / d(x, y)$.

Then T has a unique fixed point.

Proof. Let $x_0 \in X$, $x_{n+1} = T(x_n)$, $s_n = d(x_n, x_{n+1})$, $n = 0, 1, 2, \cdots$. First, we shall prove that T has a fixed point. We may assume $s_n > 0$ for each n. By (2),

(a) $s_0 s_1 = s_0 d(T(x_1), T(x_0)) \leqq \alpha_1(s_0) s_1 + {}_2(s_0) s_0 + {}_4(s_0) d(x_0, x_2) + \alpha_5(s_0) s_0$. Since $d(x_0, x_2) \leqq S_0 + s_1$, from (a)

(b) $s_1 \leqq \frac{\alpha_1(s_1) + \alpha_3(s_1) + \alpha_5(s_1)}{s_1 - \alpha_2(s_1) - \alpha_3(s_1)} s_0$, Similary

(c) $s_2 \leqq \frac{\alpha_1(s_1) + \alpha_3(s_1) + \alpha_5(s_1)}{s_1 - \alpha_2(s_1) - \alpha_3(s_1)} s_1$,

By symmetry of x, y in (2), we may assume $\alpha_1 = \alpha_2$, $\alpha_3 = \alpha_4$. From (b), (c) and induction,

(d) $s_{n+1} \leqq \alpha(s_n)$, $n = 0, 1, 2, \cdots$, where

$$\alpha(t) = \frac{\alpha_1(t) + \alpha_3(t) + \alpha_5(t)}{t - \alpha_2(t) - \alpha_4(t)} t, \quad t > 0.$$

From (1), $\alpha(t) < t$ for $t > 0$, $\{s_n\}$ is decreasing and therefore converges to some point s in $[0, \infty)$. If $s > 0$, then $s = \lim_{n \to \infty} s_{n+1} \leqq \lim_{n \to \infty} \sup \alpha(s_n)$, (e) Since α is lower semicontinuous from the right, from (e), $s \leqq \alpha(s)$, a contradiction. So, $s = 0$

Next. Suppose that $\{x_n\}$ is not a Cauchy sequence. Then exists $r > 0$ and sequences $\{p(n), q(n)\}$ such that for each $n = 0, 1, 2, \cdots$, (f) $p(n) < q(n) > n$, $d(p(n), q(n)) \geqq r$ and $d(x_{p(n)-1}, x_q(n)) < r$. Let $n \geqq 0$, $c_n = d(x_{p(n)}, x_{q(n)})$. Then $r \leqq C_n \leqq d(x_{p(n)-1}, x_{q(n)}) + d \quad d(x_{p(n)} - 1, x_{p(n)}) \leqq r + s_{p(n)-1}$

Since $\{s_n\}$ converges to 0, $\{c_n\}$ converges to r from the right.

By (2), $c_n d(T(xp_{(n)}), T(x_{q(n)})) \leqq \alpha_1(c_n)s_{p(n)} + \alpha_2(c_n)s_{q(n)} + \alpha_3(c_n)d(x_{p(n)}, x_{q(n)+1})$
$+ \alpha_4(c_n)d(x_{q(n)}, x_{p(n)+1}) + \alpha_5(c_n)c_n.$

By letting $n\to\infty$.

$r^2 \leqq (\alpha_0(r) + \alpha_4(r) + \alpha_5(r))r$ contradict to (1). Hence $\{x_n\}$ is Cauchy sequence.

Since (X, d) is complete, $\{x_n\}$ converges to some point x in X.

Since each $s_n > 0$, there exists a subsequence $\{x_{k(n)}\}$ of $\{x_n\}$ such that $x_{k(n)} \neq x$ for each n.

Let $n \geqq 0$, $d_n = (x, x_{k(n)})$. Then from (2),

$d(x_{k(n)+1}, T_{(x)}) = d(Tx_{k(n)}), T(x)) \leqq [\alpha_1(d_n)s_{k(n)} + \alpha_2(d_n)d(x, T(x)) + \alpha_3(d_n)d(x_{k(n)}, T(x))$
$+ \alpha_4(d_n)d(x, x_{k(n)+1}) + \alpha_5(d_n)d_n]/d_n.$

So $d(x, T(x)) \leqq \frac{\alpha_2(d_n) + \alpha_3(d_n)}{d_n} d(x, T(x)) + 0(n)$, where $\{0(n)\}$ coverges to O.

Since $\alpha_2(t) + \alpha_3(t) < t/2$ for $t>0$, $d(x, T(x)) \leqq d(x, T(x))/2$.

Therefore $T(x) = x$. If T has two distinct fixed points x_1, x_2, in X, then $d(x_1, x_2) = d(T(x_1), T(x_2))$ $\leqq (\alpha_3(d(x_1, x_2)) + \alpha_4(d(x_1, x_2)) + \alpha_5(d(x_1, x_2)) < d(x, x_2)$, a contradiction.

Hence T has a unique fixed point in X.

Theorem. Let (X, d) be a nonempty compact metric space.

Let T be a continuous function of X into it self. Suppose that there exists non-negative real-valued decreasing functions $\alpha_1, \cdots, \alpha_5$ on $(0, \infty)$ such that

(a) $\alpha_1 + \alpha_2 + \cdots + \alpha_5 \leqq 1$

(b) $\alpha_1 = \alpha_2$ and $\alpha_3 = \alpha_4$

(c) for any distinct x, y in X,

$d(T(x), T(y)) < a_1 d(x, T(x)) + a_2 d(y, T(y)) + a_3 d(x, T(y)) + a_4 d(y, T(x)) + a_5 d(x, y).$

where $a_i = \alpha_i(d(x, y))$.

Then T has a unique fixed point.

Proof. Let F be the function on X by $F(x) = d(x, T))$, $x \in X$. Then F is coutinuous on X. So F takes its minimum value at aome x_0 in X. We shall prove that x_0 is a fixed point of T. Suppose not. Let

$$x_1 = T(x_0), x_2 = T(x_1), \quad x_3 = T(x_2),$$
$$b_0 = d(x_0, x_1), \quad b_1 = d(x_1, x_2), \quad b_2 = d(x_2, x_3).$$

Then $b_0 > 0$, $b_1 > 0$. From (c),

(5) $(1 - \alpha_2(b_0) - \alpha_3(b_0))b_1 < (\alpha_1(b_0) + \alpha_3(b_0) + \alpha_5(b))b_0.$

Let $m(t) = 1 - \alpha_2(t - \alpha_3(t)$, $n(t) = \alpha_1(t) + \alpha_3(t) + \alpha_5(t)$, $t>0$.

From (a), $m(b_0) > 0$. So (6) $b_1 < \frac{n(b_0)}{m(b_0)} < b_0$.

Similarly,

(7) $b_2 < \frac{v(b_1)}{u(b_2)} < b_1$, where $u(t) = 1 - \alpha_1(t) - \alpha_4(t)$, $v(t) = \alpha_2(t) + \alpha_4(t) + \alpha_5(t)$, $t>0$.

From (6) and (7), $b_2 < \frac{v(b_1)n(b_0)}{u(b_1)m(b_0)} b_0.$

Let $b=\min\{b_0, b_1\}$.

Then $v(b_1)n(b_0)-u(b_1)m(b_0)\leqq v(b)n(b)-u(b)m(b)<0$,

Hence $\frac{v(b_1)n(b_0)}{u(b_1)m(b_0)}<1$. Then $b_2<b_1$, a contradiction to the minimality of b_0.

So T has a fixed point.

If x, y are distinct fixed points of T, from (c), $d(x, y)=d(T(x), T(y))<d(x, y)$. a contradiction.

Hence $x=y$.

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