韓國數學教育學會誌〈數學教育〉
1976. 12. Vol. XV No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1976 Vol. XV No. 1

CAUCHY-TYPE CRITERION FOR STOCHASTIC CONVERGENCE

by

In Whan Chung

Yonsei University, Seoul, Korea

1. DEFINITIONS AND NOTATIONS

In the present paper we shall always tacitly assume that a probability space $(\Omega, \mathfrak{U}, P)$ is given and that all random variables are defined on this space. For our further discussions, several definitions will be given first of all.

Definition (1.1). A sequence of random variables $\langle X_n\rangle$ converges almost certainly to zero if $P(\lim_{n\to\infty} X_n=0)=1$. We express this statement by the notation $\underset{n\to\infty}{\text{a.c.l.}}\ X_n=0$.

Definition (1.2). A sequence of random variables $\langle X_n\rangle$ converges almost certainly to a random variable X if the sequence $\langle X_n-X\rangle$ converges almost certainly to zero. This statement will be expressed as $\underset{n\to\infty}{\text{a.c.l.}}\ X_n=X$.

Definition (1.3). A sequence of random variables $\langle X_n\rangle$ converges in probability to a random variable X if $\lim_{n\to\infty} P(|X_n-X|>\epsilon)=0$ for any $\epsilon>0$. This statement will be expressed as $\underset{n\to\infty}{\text{Plim}}\ X_n=X$.

Definition (1.4). A sequence of random variables $\langle X_n\rangle$ converges in the r-th mean to a random variable X if all X_n and X have finite moments of order $r>0$ and if $\lim E|X_n-X|^r=0$. This statement will be expressed as $L_r-\lim_{n\to\infty} X_n=X$.

Now, consider the inequality

$$|X_j-X|\le\epsilon \tag{1.1}$$

for an arbitrary positive number ϵ, where X is a fixed random variable and $\langle X_j\rangle$ is a sequence of real random variables. Then the sets $A_{j,\epsilon}=\{w: |X_j-X|\le\epsilon\}$ and $A_{n,v}=\{w: |X_{n-v}-X_n|\le\epsilon\}$ are subsets of the sample space Ω. Define the sets

$$B_{n,\epsilon}=\bigcap_{j=n}^{\infty}A_{j,\epsilon};\ A_n^{\epsilon}=\bigcap_{v=1}^{\infty}A^{\epsilon}_{n,v};\ B_n^{\epsilon}=\bigcap_{k=n}^{\infty}A_k;$$

$$B^{\epsilon}=\bigcup_{n=1}^{\infty}B_n^{\epsilon};\ B_{\epsilon}=\bigcup_{n=1}^{\infty}B_{n,\epsilon}. \tag{1.2}$$

Then, we have $B_\epsilon=\bigcup_{n=1}^{\infty}\bigcap_{j=n}^{\infty}A_{j,\epsilon}=\lim_{j\to\infty}\text{Inf}\ A_{j,\epsilon}$, and see that the sets $A_{j,\epsilon}$, $A_{n,v}$, $B_{n,\epsilon}$, A_n^{ϵ}, B_n^{ϵ}, B^{ϵ}, and B_ϵ belong to $\mathfrak{U}$. Clearly B_ϵ is the event that inequality (1.1) holds for almost all j simultaneously. Let $<\epsilon_\nu>$ be a sequence of positive numbers such that $\lim_{\nu\to\infty} \epsilon_\nu=0$. Again we can define the set B for each and the set $B=\bigcap_{\nu=1}^{\infty} B_{\epsilon_\nu}$. Then we have again $B\in \mathfrak{U}$. The set B is the event that for any $\epsilon>0$, however small, the inequality (1.1) holds for almost all J; in other words, B is the event such that $\lim_{n\to\infty} X_n=X$. It is well known fact that if $\operatorname{Plim}_{n\to\infty} X_n=X$ and $h(X)$ is a continuous function then $\operatorname{Plim}_{n\to\infty} h(X_n)=h(X)$ and $\lim_{n\to\infty} E(h(X_n))=E(h(X))$.

2. MAIN RESULTS

In the present section, our main results will be proved.

Lemma (2.1). We have

(2.1) $$\underset{n\to\infty}{\text{a.c.l.}}\ X_n=X \text{ if, and only if } \lim_{n\to\infty} P(\sup_v|X_{n+v}-X_n|\leq\delta)=1 \text{ for every } \delta>0.$$

Proof. First of all, we prove that the condition is necessary. Since $|X_{n+v}-X_n|\leq|X_{n+v}-X|+|X_n-X|$, we see that $A_{n+v,\,\varepsilon/2}\cap A_{n,\,\varepsilon/2}\subset A^{\varepsilon}_{n,\,v}$, so that $B_{n,\,\varepsilon/2}\subset A_{n+v,\,\varepsilon/2}\cap A_{n,\,\varepsilon/2}\subset A^{\epsilon}_{n,\,v}$ for all v. Hence we have

(2.2) $$B_{n,\,\epsilon/2}\subset A_n^{\epsilon}.$$

A_n^{ϵ} is the event that the inequalities $|X_{n+v}-X_n|\leq\epsilon$ hold simultaneously for all v; in other words,

(2.3) $$A_n^{\epsilon}=\{w : \sup_{\varepsilon/2}|X_{n+v}-X_n|\leq\epsilon\}.$$

Hence we may conclude from (2.2) and (2.3) that

$$P(\sup_v|X_{n+v}-X_n|\geq P(B_{n,\,\varepsilon/2}).$$

This relation together with the inequality

$$\lim_{n\to\infty} P(B_{n,\,\varepsilon/2})=\lim_{n\to\infty} P(\bigcap_{j=n}^{\infty}\{w : |X_j-X|\leq\epsilon\}\lim_{n\to\infty} P(\underset{n\to\infty}{\text{a.c.l.}}\ X_n=X)=1,$$

obtained from the definitions (1.1) and (1.2), imply our assertion

$$\lim_{n\to\infty} P(\sup_v|X_{n+v}-X_n|\leq\delta)=1.$$

To prove the sufficiency of the condition, we note first of all that for any integer $q\geq n$

$$A_n^{\varepsilon/2}\subset\{w : |X_{q+v}-X_n|\leq\varepsilon/2\}\cap\{w : |X_q-X_n|\leq\varepsilon/2\}\subset\{w|X_{q+v}-X_q|\leq\varepsilon/2\},$$

which are equivalent to $A_n^{\varepsilon/2}\subset A^{\epsilon}_{q,\,v}$ for all $q\geq n$ and all v. Hence, we have $A_n^{\varepsilon/2}\subset A_q^{\epsilon}$ for all $q\geq n$, so that $A_n^{\varepsilon/2}\subset B_n^{\epsilon}$ and

(2.4) $$\lim_{n\to\infty} P(A_n^{\varepsilon/2})\leq\lim_{n\to\infty} P(B_n^{\epsilon})=P(B^{\epsilon}).$$

Therefore, using (2.4) and the condition $\lim_{n\to\infty} P(\sup^v|X_{n+v}-X_n|\leq\delta)=1$, we may conclude that for any fixed $\epsilon>0$

(2.5) $$P(B^{\epsilon})=1.$$

Let ϵ_ν be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_\nu=0$, and form the set $B=\bigcap_{\nu=1}^{\infty} B^{\epsilon_\nu}$. Then $P(B)=1$ from (2.5). Since the sequence $<X_j>$ satisfies the Cauchy condition on every point of the set B, there exists a function $Y=Y(w)$ such that $\lim_{n\to\infty} X_n(w)=Y(w)$ for all $w\in B$.

Extending this function to the whole space Ω by defining

$$X(w)=\begin{cases}Y(w), & \text{if } w\epsilon E\\ 0, & \text{if } w\epsilon B^C\end{cases}$$

Since $P(B^C)=0$, we see that $\underset{n\to\infty}{\text{a.c.l.}}\ X_n=X$. Q.E.D.

Lemma (2. 2). Let $<X_n>$ be a sequence which has the property that for any $\epsilon>0$, $\delta>0$ there exists an integer $N=N(\epsilon;\delta)$ such that $P(|X_n-X_m|>)\leq\delta$, provided that $m, n\geq N$. then the sequence $<X_n>$ containd a subsequenc which converges almost certainly to some random variable.

Proof. We select a sequence $<\epsilon_j>$ of decreasing positive numbers such that $\sum^{\infty}{}_{j=1}\epsilon_j<\infty$. It follows from the assumptions of the lemma that there exists for each positive integer j a number N_j such that for $m, n\geq N_j$, $P(|X_m-X_n|>\epsilon_j)<\epsilon_j$. We define a sequence $<n_j>$ of integers by setting $n_1=N_1$, $n_{j+1}=\max(n_{j+1}, N_{j+1})$ for $j=1, 2, \cdots$ Then we have $n_1<n_2<\cdots$ and

(2. 6) $$P(|X_{n_j}-X_{n_{j+1}}|>\epsilon_j)<\epsilon_j,\ (j=1, 2, \cdots).$$

Hence, we have from (2. 6) that

(2. 7) $$P(B_k)\geq 1-\sum_{j=k}^{\infty}P(A_j^C)\geq 1-\sum_{j=k}^{\infty}\epsilon_j,$$

where $A_j=\{w: |X_{n_j}-X_{n_{j+1}}|\leq\epsilon_j\}$ and $B_k=\bigcap_{j=k}^{\infty}A_j$. Let $\epsilon>0$ and $\delta>0$ be two arbitrary numbers and select k sufficiently large that $\sum_{j=k}^{\infty}\epsilon_j\leq\min(\epsilon,\delta)=\eta$. Let m be an integer such that $m\geq k$ and v be an arbitrary positive integer and put $r=m+v$. Then the event B_k implies that $\sum_{j=m}^{r-1}|X_{n_{j+1}}-X_{n_j}|\leq\sum_{j=m}^{r-1}\epsilon_j\leq\eta$. Since $|X_{n_r}-X_{n_m}|=\sum_{j=m}^{r-1}(X_{n_{j+1}}-X_{n_j})|$, we see that $B_k\subset\{w: \sup_v|X_{n_{m+v}}-X_{n_m}|\leq\epsilon\}$, provided that $m\geq k$. Therefore, using (2. 7) we have

$$P(\sup_v|X_{n_{m+v}}-X_{n_m}|\geq\epsilon)\geq P(B_k)\geq 1-\sum_{j=k}^{\infty}\epsilon_j\geq 1-\eta,$$

so that

$$P(\sup_v|X_{n_{m+v}}-X_{n_m}|>\epsilon)\leq\delta \text{ if } m\geq k.$$

This means that for any $\epsilon>0$

$$\lim_{n\to\infty}P(\sup_v|X_{n_{m+v}}-X_{n_m}|>\epsilon)=0.$$

It then follows from Lemma (2. 1) that the subsequence $<X_{n_j}>$ converges almost certainly to some andom variable. Q.E.D.

Now, we are ready to prove our main theorems, Cauchy-type criterions for convergence in probability and in r-th mean.

Theorem (2. 3). The following two statements are equivalent:

(a) $$\underset{n\to\infty}{\text{Plim}}\ X_n=X.$$

(b) It is possible to find an integer $N=N(\epsilon,\delta)$ corresponding to every $\epsilon>0$, $\delta>0$, such that

$$P(|X_n-X_m|>\epsilon)<\delta \text{ for } n, m>N.$$

Proof. Assuming the statement (a), we first prove that there exists an integer $N=N(\epsilon, \delta)$ for every $\epsilon>0$ and $\delta>0$ such that $P(|X_n-X|>\epsilon)<\delta$ for $n\geq N$. Choose integers n and m such that $n, m>N\left(\frac{\epsilon}{2}, \frac{\delta}{2}\right)$, and define the sets

$$A=\left\{w : |X_n-X|\leq\frac{\epsilon}{2}\right\}, \quad B=\left\{w : |X_m-X|\leq\frac{\epsilon}{2}\right\}, \quad C=\{w : |X_n-X_m|\leq\epsilon\}.$$

It then follows from the implication rule that

$$P(|X_n-X_m|>\epsilon)\leq P\left(|X_n-X|>\frac{\epsilon}{2}\right)+P\left(|X_m-X|>\frac{\epsilon}{2}\right)\leq\delta,$$

which implies the statement (b).

In the next, assume the statement (b). Then the sequence $\langle X_n\rangle$ satisfies the assumptions of Lemma (2.2); therefore $\langle X_n\rangle$ contains a subsequence $\langle X_{n_p}\rangle$ which converges almost certainly to some random variable X. Again, define three sets

$$S=\left\{w : |X_n-X_{n_p}|\leq\frac{\epsilon}{2}\right\}, \quad T=\left\{w : |X_{n_p}-X|\leq\frac{\epsilon}{2}\right\}, \quad U=\{w : |X_n-X|\leq\epsilon\},$$

and applying the implication rule see that

$$P(|X_n-X|>\epsilon)\leq P\left(|X_{n_p}-X|>\frac{\epsilon}{2}\right)+P\left(|X_n-X_{n_p}|>\frac{\epsilon}{2}\right).$$

If we choose $n_p>n$ and n sufficiently large, then the righthand side of the last inequality can be made arbitrary small. Hence we have the statement (a). Q.E.D.

Theorem (2.4). The following two statements are equivalent:

(a) $$L_r-\lim_{n\to\infty} X_n=X.$$

(b) It is possible to find an integer $N=N(\epsilon)$ for any $\epsilon>0$ such that $E(|X_m-X_n|^r)\leq\epsilon$ for $m, \geq N$.

Proof. First of all, assume the statement (a). Using the inequality $|a+b|^r\leq 2^r(|a|^r+|b|^r)$, we see that

$$E(|X_m-X_n|^r)=E(|(X_m-X)+(X-X_n)|^r)\leq 2^rE(|X_m-X|^r)+2^rE(|X_n-X|^r).$$

It then follows from our assumption that the right-hand side of the above inequality can be made arbitrary small by choosing m and n sufficiently large so that our statement (b) is proved.

In the next, assume the statement (b). Choose two arbitrary positive numbers ϵ and δ, and put $\epsilon_1=\epsilon\delta^r$. According to our assumption, there exists an integer $N=N(\epsilon_1)$ such that

(2.8) $$E(|X_m-X_n|^r)\leq\epsilon_1=\epsilon\delta^r \text{ for } n, m\geq N.$$

Therefore we have, for $n, m\geq N$

$$\begin{aligned}E(|X_m-X_n|^r)&=\int_{-\infty}^{\infty}|X_m-X_n|^r dF(x)\\&=\int_{|X_m-X_n|^r\geq\delta}|X_m-X_n|^r dF(x)+\int_{|X_m-X_n|^r\leq\delta}|X_m-X_n|^r dF(x)\\&\geqq\delta^r\int_{|X_m-X_n|^r\geq\delta}dF(x)=\delta^r p(|X_m-X_n|^r\geq\delta),\end{aligned}$$

provided that $F(x)$ is a distribution function. Hence, we have

$$P(|X_m-X_n|^r\geq\delta)\leq\frac{1}{\delta^r}E(|X_m-X_n|^r)\leq\frac{\epsilon_1}{\delta^r}=\epsilon,$$

from which we may conclude from Theorem (2. 3) that the sequence $\langle X_n \rangle$ converges in probability; i.e., $X = \operatorname*{Plim}_{n\to\infty} X_n$. Therefore,

$$0 = \lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = \lim_{n\to\infty} P|(X_n - X_m) - (X - X_m)| \geq \epsilon),$$

so that
$$\operatorname*{Plim}_{n\to\infty} (X_n - X_m) = X - X_m \text{ and } \operatorname*{Plim}_{n\to\infty} |X_n - X_m|^r = |X - X_m|^r.$$

Finally, we may conclude from (2. 8) that statement (a) holds. Q.E.D.

REFERENCES

1. Takeyuki Hida, Stationary Stochastic Processes, Princeton University Press, 1970.
2. Heinz Bauer, Probability Theory and Elements of Measure Theory, Holt, Rine Hart and Winston, 1968.
3. V.S. PuGachev, Theory of random Functions, Translated by O.M. Blunn, Pergmon Press, 1965.
4. J.G. Kemeny, J.L. Snell, & A.W. Knapp, Denumerable Markov Chains, D. Van Nostrand Company, 1966.