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On the sums of four squares

by

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Abstract

Lagrange proved that any positive integer is the sum of at most four squares.

We consider a elliptic function $f_\alpha(v|\tau)$ of periods 1. τ derived from θ-functions. From the important number-theoretical interpretation $\theta_3{}^4=1+8\sum_{r=1}^{\infty}q^r\sum_{\substack{m|r\\4\nmid m}}m$ we obtain $A_4(n)$ the number of representations of n as a sum of 4-squares.

Definition 1. We define the 4 following θ-functions in the notation of Tannery and Molk.

$$\theta_1(v|\tau)=-i\sum(-1)^n q^{(n+\frac{1}{2})^2}e^{(2n+1)\pi iv},$$
$$\theta_2(v|\tau)=\sum q^{(n+\frac{1}{2})^2}e^{(2n+1)\pi iv},$$
$$\theta_3(v|\tau)=\sum q^{n^2}e^{2\pi iVn},$$
$$\theta_4(v|\tau)=\sum(-1)^n q^{n^2}e^{2\pi inV},$$

where the sumation are extended over n from$-\infty$ to∞ and, $q=e^{\pi i\tau}$, im $\tau>0$, For every value of in this half-plane the functions are entire functions of v.

The zeros of $\theta_3(z:q)$ are,

$$Z=\exp\left\{2\pi i\left(n+\frac{1}{2}+\left(m+\frac{1}{2}\right)\tau\right)\right\}$$
$$=-q^{2m+1},\ m=0,\ \pm1,\pm2,\cdots\cdots$$

Since $\sum_{n=1}^{\infty}|q|^{2n-1}$ converges, we can define $F(z)$, as a function of z.

Definition 2. $F(z)=\prod_{m=1}^{\infty}(1+q^{2m-1}z)(1+q^{2m-1}z^{-1})$.

$F(z)$ is regular at all z with the exception $z=0$, and whose zeros concide with those of $\theta_3(z:q)$, $F(z)$ is a doubly periodic function without singularities, and must be a constant.

Lemma 1. $\theta_3(z:q)=\sum_{n=-\infty}^{\infty}q^{n^2}z^n=T(q)\prod_{m=1}^{\infty}(1+q^{2m-1}z)(1+q^{2m-1}z^{-1})$

where $T(q)$ is free of v and $T(q)=\prod_{n=1}^{\infty}(1-q^{2m})$

Lemma 2. From Lamma 1 we obtain

$$\theta_1'=2\pi q^{\frac{1}{4}}\prod_{m=1}^{\infty}(1-q^{2m})^3,$$

$$\theta_2=2q^{\frac{1}{4}}\prod_{m=1}^{\infty}(1-q^{2m})(1+q^{2m-1})^2,$$

$$\theta_3=\prod_{m=1}^{\infty}(1-q^{2m})(1+q^{2m-1})^2,$$

$$\theta_4=\prod_{m=1}^{\infty}(1-q^{2m})(1-q^{2m-1})^2,$$

In view of $\frac{\partial^2{}_{\mu\nu}(\nu|\tau)}{\partial\nu^2}=4\pi i\frac{\partial\theta_{\mu\nu}(\nu|\tau)}{\partial\tau}$, with $\mu=0$ or1, $\nu=0$ or 1.

Lemma 4.

$$C_{\alpha\beta}=4\pi i\left(\frac{\frac{\partial}{\partial\tau}\partial_\alpha}{\theta_\alpha}-\frac{\frac{\partial}{\partial\tau}\theta_\beta}{\theta_\beta}\right)=4\pi i\frac{\partial}{\partial\tau}\log\frac{\theta_\alpha(0|\tau)}{\theta_\beta(0|\tau)}. \quad\cdots\cdots\cdots(1)$$

Theorem. The number $A_4(n)$ of representations of a natural number n as a sum of 4 squares is given for by the formulae.

$$A_4(n) = 8\sum_{\substack{d|n \\ 4\nmid d}} d$$

(proof) By putting $v=\frac{1}{2}$ in Def 3, we have, by Lemma 3,

$$C_{42}=f_4{}^2\left(\frac{1}{2}|\tau\right)=\left(\frac{\theta_1'}{\theta_4}\ \frac{\theta_4\left(\frac{1}{2}|\tau\right)}{\theta_1\left(\frac{1}{2}|\tau\right)}\right)^2=\left(\frac{\theta_1'\theta_3}{\theta_4\theta_2}\right)^2=\pi^2\theta_3{}^4 \quad\cdots\cdots\cdots(2)$$

And finally for $v=(1+\tau)/2$

$$C_{42}=f_4{}^2\left(\frac{1+\tau}{2}\Big|\tau\right)-f_2{}^2\left(\frac{1+\tau}{2}\Big|\tau\right)=\left(\frac{\theta_1'}{\theta_4}\ \frac{\theta_2}{\theta_3}\right)^2+\left(\frac{\theta_1'}{\theta_2}\ \frac{\theta_4}{\theta_3}\right)^2 \quad\cdots\cdots\cdots(3)$$

$$=\pi^2\theta_2{}^4+\pi^2\theta_4{}^4$$

This equation together with (2) gives the important result

$$\theta_3{}^4=\theta_2{}^4+\theta_4{}^4$$

With $q=e^{\pi i\tau}$ we have

$$\frac{\partial}{\partial\tau}=\frac{\partial}{\partial q}\cdot\frac{\partial q}{\partial\tau}=\pi iq\frac{\partial}{\partial q}$$

and thus instead of (1)

$$C_{42}=-\pi^2q\frac{\partial}{\partial q}\log\frac{\theta_4(0,q)}{\theta_2(0,q)}$$

Now after (Lemma 4)

$$\frac{\theta_2(0,q)}{\theta_4(0,q)}=2q^{\frac{1}{4}}\frac{\prod_m(1+q^{2m})^2}{\prod_m(1-q^{2m-1})^2}=2q^{\frac{1}{4}}\frac{\prod_m(1-q^{4m})^2}{\prod_m(1-q^m)^2}$$

and hence

$$C_{42}=4\pi^2q\left\{\frac{1}{4q}-8\sum_{m=1}^{\infty}\frac{m8^{4m-1}}{1-q^m}+2\sum_{m=1}^{\infty}\frac{mq^{m-1}}{1-q^m}\right\}$$

Observing now, after (Def 1)

$$(\theta_3)^4=\left(\sum_{n=-\infty}^{\infty} q^{n2}\right)^4=1+8\sum_{m=1}^{\infty}\frac{mq^m}{1-q^m}-8\sum_{m=1}^{\infty}\frac{4mq^{4m}}{1-q^{4m}}$$

$$=1+8\sum_{\substack{m\geq 1\\ m\not\equiv 0 \pmod 4}}\frac{mq^m}{1-q^m}=1+8\sum_{\substack{m=\\ m\not\equiv 0 \pmod 4}}^{\infty}\sum_{k=1}^{\infty} mq^{km}$$

$$\left(\sum_{-\infty}^{\infty} q^{n2}\right)^4=1+8\sum_{r=1}^{\infty} q^r \sum_{\substack{m\mid r\\ 4\nmid m}} m$$

Lemma 3. (Euler's Identity)

$$\pi\theta_2\theta_3\theta_4=\theta_1'$$

(proof)

$$\pi\theta_1\theta_2\theta_3=\theta_1'\prod_{m=1}^{\infty}((1+q^{2m})(1+q^{2m-1})(1-q^{2m-1}))^2$$

$$=\theta_1'\prod_{m=1}^{\infty}((1+q^{2m})(1-q^{4m-2}))^2=\theta_1'H(q).$$

But

$$H(q)=\prod_{m=1}^{\infty}(1+q^{2m})(1-q^{4m-2}),$$

$$=\prod_{m=1}^{\infty}(1+q^{4m})(1+q^{4m-2})(1-q^{4m-2}),$$

$$=\prod_{m=1}^{\infty}(1+q^{4m})(1-q^{8m-4}),$$

and

$$H(q)=H(q^2).$$

Since $H(q)$ is continuous in q, and $q^{4k}\to 0$ as $k\to\infty$, we have $H(q)=H(0)=1.\ \pi\theta_2\theta_3\theta_4=\theta_1'$.

Definition 3.

$$f_\alpha(\nu|\tau)=\frac{\theta_1'}{\theta_\alpha}\ \frac{\theta(\nu_\alpha|\tau)}{\theta_1(\nu|\tau)},\quad \alpha=2,3,4.$$

Thus their squares $f_\alpha(\nu|\tau)$ are doubly periodic meromorphic functions, in other words elliptic functions of periods 1. They all have poles of 2nd order at $v=0$. They are even functions, so that their residue is 0 at these poles. Their Laurent expansions begin in all cases with $1/\nu^2$. Therefore any difference

$$f_\alpha^2(\nu|\tau)-f_\beta^2(\nu|\tau)$$

is doubly periodic, free of poles, and thus a constant. Let us put

$$\int_\alpha^2(\nu|\tau)-f_\beta^2(\nu|\tau)=C_{\alpha\beta}$$

and determine this constant; Since $\theta_\alpha(v)$ is even, $\alpha=2,3,4$, and $\theta_1(v)$ odd we have

$$f_\alpha^2(\nu|\tau)=\frac{\theta_1'}{\theta_\alpha}\left(\frac{\theta_\alpha+\theta_\alpha''\dfrac{v^2}{2}+\cdots\cdots}{\theta_1'\nu+\theta_1'''\dfrac{v^3}{6}+\cdots\cdots}\right)^2=\frac{1}{\nu^2}\ \frac{1+\dfrac{\theta_\alpha''}{\theta_\alpha}\nu^2+\cdots\cdots\cdots\cdots}{1+\dfrac{\theta_1'''}{\theta_1}\ \dfrac{v^2}{3}+\cdots\cdots}$$

$$=\frac{1}{\nu^2}\left(1+\left(\frac{\theta_\alpha''}{\theta_\alpha}-\frac{1}{3}\ \frac{\theta_1'''}{\theta_1'}\right)\nu^2+\cdots\cdots\cdots\right)$$

and

$$C_{\alpha\beta}=\frac{\theta_\alpha''}{\theta_\alpha}-\frac{\theta_\beta''}{\theta_\beta}$$

where the dash refers to differentiation with respect to v, e.g, On both sides we have power series in q, convergent for $|q|<1$. The coefficients must agree. If we write the left-hand member as

$$\sum_{\substack{n1,n2,n3,n4\\ -\infty}}^{\infty} q^{n_1^2+n_2^2+n_3^2+n_4^2}=\sum_{n=0}^{\infty}A_4(n)q^n$$

then A(n) gives the number of representations of n as a sum of 4 squares, where representations are counted separately if they differ in the arrangement of the summands, and also(n) and (−n) have to be counted as different summand.

Corollary

If n is even then

$$\sum_{\substack{d|n\\ 4\nmid d}} d=\sum_{\substack{d|n\\ d:\text{odd}}} d+2\sum_{\substack{d|n\\ d:\text{odd}}} d=3\sum_{\substack{d|n\\ d:\text{odd}}} d.$$