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Some properties of a pseudometric on the set of sub σ-fields of a σ-field $\mathfrak{F}$

by

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1. A theorem on the transformation of D into the metric

Edward. S. Boylan (1) defined a pseudometric D on the set of sub-σ-fields of a σ-field $\mathfrak{F}$ in order to prove a sufficient condition of the equiconvergence of martingales. We find in the concluding words of the paper (1) the following remark; D may well be worthy of interest in its own right and hopefully will be of value in investigating other probability questions. The purpose of this paper is to discuss some properties of D. The definition of D in (1) was as follows;

Concerning the probability space $(X. \mathfrak{F}. P)$,

we define $d(F, \mathfrak{F}') = \inf_{F' \in \mathfrak{F}'} P(F \varDelta F')$, where $F \in \mathfrak{F}$, $F' \in \mathfrak{F}'$ and $\mathfrak{F}'$ is a sub-σ-field of $\mathfrak{F}$.

Again we define $d(\mathfrak{F}_1, \mathfrak{F}_2) = \operatorname{sub}_{F_1 \in \mathfrak{F}_1} d(F_1, \mathfrak{F}_2)$, where $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are two sub-σ-fields of $\mathfrak{F}$,

and lastly we define $D(\mathfrak{F}_1, \mathfrak{F}_2) = d(\mathfrak{F}_1, \mathfrak{F}_2) + d(\mathfrak{F}_2, \mathfrak{F}_1)$,

then D is a pseudometric on the set of sub-σ-fields of $\mathfrak{F}$, (See theorem 1 of (1))

We shall show that the pseudometric D can be transformed into the metric.

In order to do this, we shall introduce one more pseudometric δ in a σ-field $\mathfrak{F}$ as follows;

$$\delta(A_1, A_2) = p(A_1 \varDelta A_2) \quad (A_1, A_2 \in \mathfrak{F})$$

It is suggested in J. Neveu (2) that δ is a pseudometric and $\mathfrak{F}/p$ forms a field. We shall give here the complete proof of this suggestion.

The relation $A_1 \overset{=}{p} A_2$,, if $\delta(A_1, A_2) = 0$ $(A_1, A_2 \in \mathfrak{F})$, is a equivalence relation in $\mathfrak{F}$.

Lemma 1. $\mathring{\mathfrak{F}} = \mathfrak{F}/p$ is a σ-field is $\mathfrak{F}$ if a σ-field.

Proof: 1) $A_1 \overset{=}{p} A_2 \Leftrightarrow p(A_1 \varDelta A_2) = 0 \Leftrightarrow p(A_1^c \varDelta A_2^c) = 0 \Leftrightarrow A_1^c \overset{=}{p} A_2^c$.

$$\begin{aligned}(\mathring{A}_1^c) &= \{A; A \overset{=}{p} A_1^c\} \\ &= \{A; A^c \overset{=}{p} A_1\} \\ &= \{A; A^c \in \mathring{A}_1\}\end{aligned}$$

Therefore we define $(\mathring{A}_1^c) = (\mathring{A}_1)^c$. Therefore $(\mathring{A}_1)^c \in \mathring{\mathfrak{F}}$.

This definition does not depend on A_1, but only to $\mathring{A}_1$, that is, $(\mathring{A}_1)^C$ has a meaning.
Thus we have proved that $\mathring{\mathfrak{F}}$ is closed under the complementation.

2. $(A_1 \cup A_2)^0 = \{A; A \bar{p} A_1 \cup A_2\}$, ... (1)

$(A_1 \cup A_2) \Delta A_1 = A_2 - A_1$, ... (2)

$(A_1 \cup A_2) \Delta A_2 = A_1 - A_2$, ... (3)

$A_1 \Delta A_2 = (A_2 - A_1) \cup (A_1 - A_2)_0$. .... (4)

When $\mathring{A}_1$ and $\mathring{A}_2$ are different classes of p-equivalence, we have $p(A_1 \Delta A_2) \neq 0$,
i,e, $p(A_1 - A_2) + p(A_2 - A_1) \neq 0$,
that is, at least one of the two terms of the lefthand side is not zero.
Therefore, from (2), (3) and (4), we can say that at least one of A_1 and A_2 is not p-equivalent with $A_1 \cup A_2$.
If $A_1' \in \mathring{A}_1$, $A_2' \in \mathring{A}_2$,
then $p[(A'_1 \cup A_2') \Delta (A_1 \cup A_2)] \leqq P(A_1' - A_1) + p(A_2' - A_2) + p(A_1 - A_1') + p(A_2 - A_2') = 0$,
i.e. $A_1' \cup A_2' \in (A_1 \cup A_2)^0$,
which means that the definition of $A_1^0 \cup A_2^0 = (A_1 \cup A_2)^0$ does neither depend on A_1 nor on A_2 but only to $\mathring{A}_1$ and $\mathring{A}_2$.
Therefore we can define $(A_1 \cup A_2)^0 = A_1^0 \cup \mathring{A}_2$ Therefore $\mathring{A}_1 \cup \mathring{A}_2 \in \mathfrak{F}$.

This proof can be extended to an arbitrary countable union $\bigcup_{n=1}^{\infty} \mathring{A}_n = \left(\bigcup_{n=1}^{\infty} A_n\right)^0$

Lemma 2. $\mathring{\mathfrak{F}}_1 = \mathfrak{F}_1/p$ is a sub-σ-field of $\mathring{\mathfrak{F}} = \mathfrak{F}/p$ if $\mathfrak{F}_1$ is a sub-σ-field of a σ-field $\mathfrak{F}$.

Proof: From lemma 1, we know that $\mathfrak{F}_1/p$ and $\mathfrak{F}/p$ are σ-fields because $\mathfrak{F}_1$ and $\mathfrak{F}$ are σ-fields.
We have only to show that any member $\mathring{A}_1$ of $\mathfrak{F}_1/p$ is also a member of $\mathfrak{F}/p$.
Now $\mathring{A}_1 \in \mathfrak{F}_1/p$ means $\mathring{A}_1 = \{A; A \bar{p} A_1, A \in \mathfrak{F}_1, A_1 \in \mathfrak{F}_1\}$.
As $\mathfrak{F}_1 \subset \mathfrak{F}$, the A, A_1 belong to $\mathfrak{F}$.
Therfore $\mathring{A}_1$ can be assumed to be identical with $\{A; A \bar{p} A_1, A \in \mathfrak{F}, A_1 \in \mathfrak{F}\}$, which is a member of $\mathfrak{F}/p$.

Theorem 1. The pseudometric D can be transformed into the metric.

Proof: We define a probability $\mathring{P}$ on $\mathring{\mathfrak{F}} = \mathfrak{F}/p$ such that $\mathring{p}(\mathring{A}) = p(A_1)$, $(A_1 \in \mathring{A}_1)$, $\mathring{\delta}(\mathring{A}_1, \mathring{A}_2) = \mathring{P}(\mathring{A}_1 \Delta \mathring{A}_2) = P(A_1 \Delta A_2)$, $(\mathring{A}_1, \mathring{A}_2 \in \mathring{\mathfrak{F}}, A_1, A_2 \in \mathfrak{F}, A_1 \in \mathring{A}_1, A_2 \in \mathring{A}_2)$. $\mathring{\delta}$ defines a metric on $\mathring{\mathfrak{F}}$ because all nonempty subsets of P-measure zero in $\mathring{\mathfrak{F}}$ are transformed into the empty set $\mathring{\phi}$ of $\mathring{\mathfrak{F}}$.
We shall prove that on the set of sub-σ-fields of $\mathring{\mathfrak{F}}$, D becomes a metric $\mathring{D}$ which is defined as follows;

$$\mathring{D}(\mathring{\mathfrak{F}}_1, \mathring{\mathfrak{F}}_2) = D(\mathfrak{F}_1, \mathfrak{F}_2) \text{ where } \mathring{\mathfrak{F}}_1 = \mathfrak{F}_1/p,\ \mathring{\mathfrak{F}}_2 = \mathfrak{F}_2/p$$

We have only to prove that if $\mathring{D}(\mathring{\mathfrak{F}}_1, \mathring{\mathfrak{F}}_2) = 0$, then $\mathring{\mathfrak{F}}_1 = \mathring{\mathfrak{F}}_2$ ($\mathring{\mathfrak{F}}_1, \mathring{\mathfrak{F}}_2$ are sub-σ-fields of $\mathring{\mathfrak{F}}$).
By the corollary 1 of (1), from $\mathring{D}(\mathring{\mathfrak{F}}_1, \mathring{\mathfrak{F}}_2) = 0$, it follows that every set $\mathring{F}_1$ in $\mathring{\mathfrak{F}}_1$ differs from a set $\mathring{F}_2$ in $\mathring{\mathfrak{F}}_2$ by at most a set of measure zero.
But the set of measure zero in $\mathring{\mathfrak{F}}_1$ was only the empty set $\mathring{\phi}$.

We can say that $\mathring{F}_1$ is identical with $\mathring{F}_2$, in other words, $\mathring{\mathfrak{F}}_1=\mathring{\mathfrak{F}}_2$

2. A problem of convergence of D to zero

As E.S. Boylan (1) p. 556, remarked, through examples, that it is possible for $\mathfrak{F}_n$ to increase or decrease to $\mathfrak{F}_\infty$ without $D(\mathfrak{F}_n, \mathfrak{F}_\infty)$ approaching zero.

In this section we shall research the behavior of the value of $D(\mathfrak{N}_N, \mathfrak{B}_\infty)$, when $N\to\infty$, where we use the notation of J. Neveu (2) proposition IV-4-3, that is, $\{\mathfrak{B}_n, n\geq 1\}$ are independent sub-σ-algebras of a given σ-algebra $\mathfrak{N}$, $\mathfrak{N}_N$ is the σ-algebra generated by $\mathfrak{B}_n$ $(n\geq N)$ and $\mathfrak{B}_\infty$ is the σ-algebra of terminal (Asymptotic) events.

Then $\mathfrak{N}_N$ decrease as $N\to\infty$, and by the definition $\bigcap_{N=1}^{\infty}\mathfrak{N}_N=\mathfrak{B}_\infty$. (0, 1)-law shows that if we neglect the sets of probability zero, $\mathfrak{B}_\infty=\{\phi, \Omega\}$. To evaluate $D(\mathfrak{N}_N, \mathfrak{B}_\infty)$, let a set $A_N(\in\mathfrak{N}_N)$ be chosen, $d(A_N, \mathfrak{B}_\infty)=\inf\{p(A_N), p(A_N^C)\}$, for every $A_N\in\mathfrak{N}_N$, whose probability is nonzero.

Let

$$A_N'=\begin{cases} A_N^C \text{ when } P(A_N)\geq P(A_N^C) \\ A_N \text{ when } P(A_N)<P(A_N^C). \end{cases}$$

Then $D(\mathfrak{N}_N, \mathfrak{B}_\infty)=\sup\{p(A_N'), A_N'\in\mathfrak{N}_N\}$.

There exists $\mathring{A}_N\in\mathfrak{N}_N$ such that $D(\mathfrak{N}_N, \mathfrak{B}_\infty)\geq p(\mathring{A}_N)\geq D(\mathfrak{N}_N, \mathfrak{B}_\infty)-\varepsilon$, for all $\varepsilon>0$.

Therefore we can say as follows;

If $p(\mathring{A}_N)\downarrow 0$, then $D(\mathfrak{N}_N, \mathfrak{B}_\infty)$ approaches zero as $N\to\infty$, but if $p(\mathring{A}_N)\downarrow C>0$, $(N\to\infty)$ for a certain constant C, then $D(\mathfrak{N}_N, \mathfrak{B}^\infty)$ does not approach zero as $N\to\infty$, even though $D(\mathfrak{N}_N, \mathfrak{B}_\infty)$ decreases monotonically.

Reference

(1) Edward. S. Boylan (1971) The Annals of Mathematical Statistics Vol. 42. No.2 -April 1971.

(2) J. Neveu (1965) Mathematical foundations of the calculus of probability Holden-Day San francisco.