Global Theory of Einstein-Cartan Equations
—Gödel Universe with Torsion—

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Abstract

Gödel models of the universe filled with fluid are studied in the framework of the Einstein-Cartan theory of gravitation. It is assumed that the models admit a group of motions simply transitive on space-time. The combined effects of spin and rotation (vorticity) are studied with a particular attention to whether the field equations impose any restriction on alignement of spin direction (a polarized spin distribution). The solutions are found explicitly in a closed form, which show that spin components are vanishingly small except in the direction of z-axis (the compass of inertia) in which they can assume an arbitrary distribution.

I. Introduction

The Einstein-Cartan theory of gravitation accepts as a model of space-time, a non-Riemannian four dimensional differential manifold with a metric tensor and a linear connection compatible with the metric.

The torsion of space-time is related to the spin of matter in such a way that the field equations in a vacuum remain the same as in the classical general relativity. A present state of the theory is presented recently in a review article of Hehl et. al. [1]. Cosmological models with torsion were first studied in the hope that the singularities so ubiquitous in the solutions of Einstein's equations might be averted.

Kopczynski [2] found the first non-singular cosmological models with spin, where the metric is Friedmann-like. Physical properties of the non-singular universes were examined by Trautman [3] and later by Stewart and Hajicek [4]. Further Tafel [5] obtained the complete classification of Bianchi type homogeneous (spatially) model with torsion.

These examples show that for a cosmological model endowed with polarized (as opposed to random) spin, the distribution of spin strongly determines the allowable symmetries of the metric tensor, vice versa. Clearly, then, a study of the compatibility between metric and spin that is exacted by the field equations is prerequisite to a systematic study of cosmological models with torsion.

However, up to now, all papers on studies of these conditions are for non-rotating (non off-diagonal metric) models. The importance of rotating effect can be tens of orders of magnitude greater than the spin [1]. Therefore we start our study on the simplest cosmological model with rotation i.e. Gödel universe.

We found that the spin polarization effect is striking and that only the one direction is allowed for the alignement of spin.

II. Field Equations

In this theory, gravitation field is described by two tensor fields, namely the metric g_{ij} and the torsion $Q^i{}_{jk}$, where $Q^i{}_{jk} = \Gamma^i{}_{kj} - \Gamma^i{}_{jk}$ ($\Gamma^i{}_{jk}$ are coefficients of linear, metric connection Γ with respect to a holonomic frame). The Einstein tensor of the connection Γ is proportional to the canonical energy-momentum tensor of matter $t^i{}_j$.

$$R^i{}_j - \tfrac{1}{2}\delta^i{}_j R = 8\pi G/c^4 \; t^i{}_j. \tag{2.1}$$

The torsion tensor Γ is determined by the spin density tensor of matter $S^i{}_{jk}$

$$Q^i{}_{jk} = 8\pi G/c^3 \; (S^i{}_{jk} + \tfrac{1}{2}\delta^i{}_j S_k - \tfrac{1}{2}\delta^i{}_k S_j),$$
$$S_i = S^j{}_{ij}. \tag{2.2}$$

Taking into account the metric condition imposed on Γ and equations (2.2) one may express the connection coefficients by the Christoffel symbols and the spin tensor.

$$\Gamma^i{}_{jk} = \{^{\,i}_{jk}\} - 4\pi G/c^3 (S^i{}_{jk} + S_{jk}{}^i + S_{kj}{}^i - \delta^i_k S_j$$
$$+ g_{jk} S^i). \tag{2.3}$$

If we substitute (2.3) into equation (2.1), we obtain the single-written equation

$$\tilde{R}_{ij} - \tfrac{1}{2} g_{ij}\tilde{R} = 8\pi G/c^4 t_{ij} + 4\pi G/c^4 \tilde{\nabla}_k (S^k{}_{ij}$$
$$+ S_{ij}{}^k + S_{ji}{}^k) \; - (4\pi G/c^3)^2$$
$$[2S_{ijk}S^k + 2S_{ikm}S^{mk}{}_j + S_{ikm}S_j{}^{km}$$
$$+ g_{ij}(S_k S^k - S_{klm}S^{mlk} - \tfrac{1}{2} S_{klm}$$
$$S^{klm})], \tag{2.4}$$

instead of the system of equations (2.1) and (2.2). The symbol $\sim$ denotes objects related to the Riemannian connection associated with the metric tensor g_{ij}.

Since, due to equation (2.4), the torsion was eliminated, one may use this equation to compare the Einstein-Cartan theory with the classical theory of gravitation. One sees that the metric of space-time depends not only on the energy momentum distribution but also on spin distribution.

Let us now apply equation (2.4) to estimate the influence of spin in the case of the Weyssenhoff fluid, its matter tensors being defined by

$$t^k{}_i = u^k h_i - p\delta^k{}_i,$$
$$S^k{}_{ij} = u^k S_{ij}, \quad u^k S_{kj} = 0.$$

In the formula the vector field u^i is the velocity vector of the fluid, h^i is the vector of its enthalpy density, p is its pressure, while S_{ij} is the tensor of spin in the matter rest-frame. One can write the vector of enthalpy density in the form

$$h_i = (\rho + p) u_i + c u^j u^k \tilde{\nabla}_k S_{ij},$$

where $\rho = u^i u^j t_{ij}$ is energy density in the matter rest-frame.

In the present case equation (2.4) takes on the form

$$\tilde{R}_{ij} - \tfrac{1}{2} g_{ij}\tilde{R} = 8\pi G/c^4 [(\rho + p - 4\pi G/c^2 S^2) u_i u_j$$
$$- (p - 2\pi G/c^2 S^2) g_{ij} - c(g^{kl}$$
$$+ u^k u^l) \tilde{\nabla}_k S_{l(i} u_{j)}]. \tag{2.5}$$

where $S^2 = \tfrac{1}{2} S_{ij} S^{ij}$. One sees that the square term in spin appearing in the above equation contributes to the effective energy density and pressure

$$\bar{\rho} = \rho_{eff} = \rho - 2\pi G/c^2 \; S^2$$
$$\bar{p} = p_{eff} = p - 2\pi G/c^2 \; S^2.$$

The square term in spin behaves as an effective repulsive force. The repulsion can become important if the quantity $2\pi G/c^2 S^2$ is of the same order as the energy density. The term $c u^k u^l \tilde{\nabla}_k S_{l(i} u_{j)}$ is interpreted as u^i orthogonal energy flux, arising from the exchange of spin angular momentum and orbital angular momentum. Also noting in equation (2.4) the left hand side is symmetric in indices (i,j), its antisymmetric part in the right hand side expression must be zero.

We get, therefore, the spin conservation equation

$$\tilde{\nabla}_k S_{(ij)}{}^k = -[u_j u^l \tilde{\nabla}_k (S_{il} u^k) - u_i u^l \tilde{\nabla}_k (S_{jl} u^k)]. \quad (2.6)$$

III. Gödel Universe with Torsion

We assume that there exists a group of motions simply transitive on spacetime.

Gödel [6] showed that the line element

$$ds^2 = (dt + e^{ax} dy)^2 - [(dx)^2 + 1/2 e^{2ax} (dy)^2 + (dz)^2], \quad (3.1)$$

where a is a constant, is compatible with our assumption and admits a four-parameter group of isometry.

$$t' = t + t_o,$$
$$x' = x + x_o,$$
$$y' = y e^{-x_o a} + y_o,$$
$$z' = z + z_o,$$

where t_o, x_o, y_o, z_o are the parameter of groups.

Ricci tensor is found to be

$$R^{ij} = a^2 \delta^i{}_o \delta^j{}_o, \quad (3.2)$$

and scalar curvature is $\tilde{R} = \tilde{R}^{ij} g_{ij} = a^2$.

The contravariant metric components are

$$g^{ij} = \begin{pmatrix} -1 & 0 & 2e^{-ax} & 0 \\ 0 & -1 & 0 & 0 \\ 2e^{-ax} & 0 & -2e^{-2ax} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

If we define $G^{ij} = R^{ij} - \frac{1}{2} g^{ij} R$, Einstein tensor becomes, in unit $c=1$,

$$G^{ij} = (\bar{\rho} + \bar{p}) u^i u^j - \bar{p} g^{ij} - 2(u_m u^k + \delta^k{}_m) \tilde{\nabla}_k S^{m(i} u^{j)}. \quad (3.3)$$

We introduce the time-like velocity vector u^i such that

$$u^i = \delta^i_o \text{ and } u_i = g_{ij} u^j = g_{oi}, \ u_i u^i = 1.$$

Since we know the metric, we can easily calculate the Riemannian covariant derivative $\tilde{\nabla}_k S^{m(i} u^{j)}$. The left hand side is already known. Therefore the symmetric part of the field equations becomes, after some calculation, as follows.

The diagonal parts are

$$G^{00} = \frac{3}{2} a^2 = \bar{\rho} + 2\bar{p} - 2\left[\frac{\partial}{\partial x^\mu} S^{\mu o} + e^{ax} \dot{S}^{20} + a e^{ax} S^{21} \right].$$
$$G^{11} = G^{33} = \bar{p} = \tfrac{1}{2} a^2,$$
$$G^{22} = a^2 e^{-2ax} = 2e^{-2ax} \bar{p},$$

where, greek indices μ, ν run only 1, 2, 3. Therefore we get

$$\frac{1}{2} a^2 = \bar{\rho} - 2\left[\frac{\partial}{\partial x^\mu} S^{\mu 0} + e^{ax} S^{20} + a e^{ax} S^{21} \right], \quad (3.3)$$

$$\bar{p} = \tfrac{1}{2} a^2 \quad (3.4)$$

The off-diagonal parts are

$$G^{01} = \frac{\partial}{\partial x^\nu} S^{\nu 1} + \dot{S}^{01} + e^{ax} \dot{S}^{21} + \frac{2}{a} e^{ax} S^{02} = 0, \quad (3.5)$$

$$G^{02} = \frac{\partial}{\partial x^\nu} S^{\nu 2} + \dot{S}^{02} - a(e^{-ax} S^{01} + S^{21}) = 0, \quad (3.6)$$

$$G^{03} = \frac{\partial}{\partial x^\nu} S^{\nu 3} + \dot{S}^{03} + \dot{S}^{23} e^{ax} = 0. \quad (3.8)$$

$G^{\mu\nu} = 0$ for $\mu, \nu = 1, 2, 3$ and $\mu \neq \nu$.

The antisymmetric part of field equation (2.6), i.e. the spin conservation equation gives us

$$\left.\begin{aligned} &\dot{S}^{12} = 0 && (3.8) \\ &\dot{S}^{23} - a e^{-ax} S^{13} = 0 && (3.9) \\ &\dot{S}^{31} + \frac{a}{2} e^{ax} S^{23} = 0 && (3.10) \end{aligned}\right\} \quad (A)$$

Further, from the side condition $S_{ma} u^a = 0$, we get $S^{ma} u_a = 0$, this implies $S^{ma} g_{ao} = 0$. and we get

$$S^{10}+S^{12}g_{02}=0, \qquad S^{10}+S^{12}e^{ax}=0 \tag{3.11}$$

$$S^{20}=0 \tag{3.12}$$

$$S^{30}+S^{32}g_{02}=0, \qquad S^{30}+S^{32}e^{ax}=0 \tag{3.13}$$

Using (3.11)−(3.13), we eliminate $S^{\mu o}$ components ($\mu=1,2,3$) from the set of field equations (3.3)−(3.7).

$$\tfrac{1}{2}a^2=\rho-2\left[2ae^{ax}S^{21}-e^{ax}\left(\frac{\partial}{\partial x}S^{12}+\frac{\partial}{\partial z}S^{32}\right)\right], \tag{3.3'}$$

$$\frac{\partial}{\partial x^\nu}S^{\nu 1}=0, \tag{3.5'}$$

$$\frac{\partial}{\partial x^\nu}S^{\nu 2}=0, \tag{3.6'}$$

$$\frac{\partial}{\partial x^\nu}S^{\nu 3}=0. \tag{3.7'}$$

Using the equation (3.6)′, we get finally the following form of set of equations.

$$\rho=\frac{1}{2}a^2+4e^{ax}S^{21}, \tag{3.14}$$

$$\frac{\partial}{\partial y}S^{21}+\frac{\partial}{\partial z}S^{31}=0, \tag{3.15}$$

$$\frac{\partial}{\partial x}S^{13}+\frac{\partial}{\partial y}S^{23}=0, \tag{3.16}$$

$$\frac{\partial}{\partial x}S^{12}+\frac{\partial}{\partial z}S^{32}=0. \tag{3.17}$$

(B)

Notice if we define $S=(S^{23}, S^{31}, S^{12})$, the set of equations (3.15—17), can be easily ∇ written in the form $\nabla\times\mathbf{S}=0$, therefore there exists some scalar function ψ such that $\mathbf{S}$ are the gradient of ψ.

The set (3.8−10) can be easily solved in the closed form. However, due to the fact that the set (B) determines only the spatial part of spin density components and the set (A), on the contrary, determines the time dependent part for each $\mathbf{S}$, we must solve these coupled set (A) and (B) simultaneously. This allows us only the narrow class of the possibility of solutions which show a higher degree of spin polarization.

The set (A) permits the following solutions

$$S^{12}=f(\mathbf{x})$$

$$S^{31}=A(\mathbf{x})\cos\frac{a}{\sqrt{2}}t+B(\mathbf{x})\sin\frac{a}{\sqrt{2}}t$$

$$S^{23}=2e^{-ax}\left[B(\mathbf{x})\cos\frac{a}{\sqrt{2}}t-A(\mathbf{x})\sin\frac{a}{\sqrt{2}}t\right],$$

where $\mathbf{x}$ stands for x,y,z and, f,A,B are an arbitrary function of $\mathbf{x}$. However, the equations (3.14) and (3.17) allow us that f is the function of z alone, and A,B are the function of x, and y. From the equation (3.16), we conclude that

$$\frac{\partial}{\partial x}A(x,y)=-\sqrt{2}\,e^{-ax}\frac{\partial}{\partial y}B(x,y)$$

$$\frac{\partial}{\partial x}B(x,y)=\sqrt{2}\,e^{-ax}\frac{\partial}{\partial y}A(x,y)$$

Let us introduce the new variable $\xi=-\frac{\sqrt{2}}{a}e^{-ax}$, then the above equations are reduced to

$$\frac{\partial}{\partial \xi}A(\xi,y)=-\frac{\partial}{\partial y}B(\xi,y)$$

$$\frac{\partial}{\partial \xi}B(\xi,y)=\frac{\partial}{\partial y}A(\xi,y)$$

These equations are nothing but Cauchy-Riemann equations for $F(\xi+iy)=B(\xi,y)+iA(\xi,y)$, where $B(\xi,y)$, $A(\xi,y)$ are a real and imaginary part of $F(\xi+iy)$ respectively. It is easily seen that $A(\xi,y)$, $B(\xi,y)$ are a harmonic function in plane of ξ,y such that they separately satisfy two dimensional Laplace equation. Therefore the whole class of solution exists involving three arbitrary function, since $A(\xi,y)$ and $B(\xi,y)$ are harmonic conjugate function of each other.

The complete determination for spin density function is the following set

$$S^{12}=f(z)$$

$$S^{31}=A(\xi,y)\cos\frac{a}{\sqrt{2}}t+B(\xi,y)\sin\frac{a}{\sqrt{2}}t$$

$$S^{23}=2e^{-ax}\left[B(\xi,y)\cos\frac{a}{\sqrt{2}}t-A(\xi,y)\sin\frac{a}{\sqrt{2}}t\right].$$

It is more natural to use the variable $\xi = -\frac{\sqrt{2}}{a} e^{-ax}$ in argument of $A(x,y)$, $B(x,y)$. The simple example of harmonic conjugate function A,B are the following

$$A = \left(\frac{-\sqrt{2}}{a} e^{-ax}\right)^2 - y^2, \quad B = -\frac{2\sqrt{2} e^{-ax}}{a} y.$$

Since $a>0$, the spatial dependent part of spin amplitude are rapidly damped out in the x direction for S^{23} and S^{31}, and also the time dependent parts for large t are highly oscillatory, therefore it may be averaged out to be zero. Anyway it shows a high degree of spin polarization in "the direction of z". From the above argument, it is likely that $S^{31}=S^{23}=0$ and the only non-vanishing spin component may be taken as S^{12} for a good approximation.

Since $S^{ij}=S_{ab}g^{ai}g^{bj}$.

$$\begin{aligned}
&S^{10}=q_1+2S_3e^{-ax}=2S_3e^{-ax},\\
&S^{20}=2q_2e^{-ax}=0,\\
&S^{30}=q_3-2S_1e^{-ax}=-2S_1e^{-ax},\\
&S^{12}=2S_3e^{-2ax},\\
&S^{23}=2S_1e^{-2ax},\\
&S^{31}=S_{31}=S_2,
\end{aligned}$$

where $q_1=S_{10}$, $q_2=S_{20}$, $q_3=S_{30}$, $S_1=S_{23}$, $S_2=S_{21}$, $S_3=S_{12}$.

By the side condition $S_{ab}u^b=0$, we have $q_1=q_2=q_3=0$

$$\begin{aligned}
&S_1=\frac{1}{\sqrt{2}}e^{ax}\left[B(\xi,y)\cos\frac{a}{\sqrt{2}}t-A(\xi,y)\sin\frac{a}{\sqrt{2}}t\right],\\
&S_2=A(\xi,y)\cos\frac{\sqrt{2}}{a}t+B(\xi,y)\sin\frac{\sqrt{2}}{a}t,\\
&S_3=\frac{1}{2}e^{2ax}f(z), \qquad\qquad (3.18)\\
&\rho=a\left(\frac{a}{2}-4e^{ax}f(z)\right)+\frac{2\pi G}{c^2}S^2,\\
&p=\frac{1}{2}a^2+\frac{2\pi G}{c^2}S^2,\\
&S^2=\frac{1}{2}e^{2ax}[f(z)]^2+A^2(\xi,y)+B^2(\xi,y).
\end{aligned}$$

The equations (3.18) are the complete set of analytic solution, which describe the Gödel universe with torsion.

Even though the metric is started with the spatial-temporal homogeneity of universe, ρ and p are both the complicated function of spatial variables x,y,z, due to the term $S^2=\frac{1}{2}S_{ij}S^{ij}$. Notice, however, p and ρ are time independent and even spatially anisotropic due to the complex interaction of spin and vortex motion. This is the completely unexpected result, (contrary to the classical general relativity expectation), whose meaning and physical interpretation is curious and unclear.

Finally, it is well known that in the Gödel universe the world-lines of the matter are in absolute rotation with angular velocity $\sqrt{4\pi G\rho}$ relative to a local inertial frame. We can generalize the above result easily by introducing the following new vector

$$\Omega_i=\tilde{\Omega}_i+1/2S_i,$$

where $\tilde{\Omega}_i=c\eta_{ijkl}u^ju^{k;l}$(the covariant derivative here is taken with respect to $\{\ \}$ is usual rotation vector associated with the local rest frame of matter and $S_a=\frac{1}{2}\eta_{abcd}u^bS^{cd}$ spin vector and η_{abcd} are a completely antisymmetric Levi-Civita tensor density. If we interpret $\tilde{\Omega}_i$ as usual way [7], $\tilde{\Omega}_i$ characterizes a uniform rotation about Z-axis, therefore since in the same approximation, as we found, $S_3=S^{12}$ is only nonvanishing, Ω_i has a large component only in Z-direction, adding $\tilde{\Omega}_3+S_3$, showing a spin alignement in Z-direction. However, it is curious fact that S_3 is an arbitrary function of Z, which does not restrict its functional form, therefore we must have some boundary conditions on infinity as to restrict its functional form.

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