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SOME STRUCTURES INDUCED ON SUBMANIFOLDS

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§ 0. Introduction.

In the present paper, we investigate some structures induced on submanifolds. In §1, we show that a submanifold M^{2n+1} of codimension 3 in a $(2n+4)$-dimensional Euclidean space admits so-called $(f, g, u, v, w, \lambda, \mu, \nu)$-structure. In § 2, we get some sufficient conditions that an odd-dimensional Riemannian manifold M^{2n+1} admits a metric f-structure with complemented frames. In § 3, we show that there exists a submanifold M^{2n+1} admitting a metric f-structure with complemented frames in a $(2n+3)$-dimensional Sasakian manifold S^{2n+3}. In § 4, 5, 6, we consider invariant hypersurfaces M^{2n} of M^{2n+1} discussed in § 3 and we show that there exists an almost contact hypersurface M^{2n-1} of M^{2n} such that $M^{2n}=M^{2n-1}\times M^1$. In § 7, we show that M^{2n-1} introduced in § 5 admits a structure which is closely related to a Sasakian structure, and we determine the form of the curvature tensor of M^{2n-1} in order that the C-holomorphic sectional curvature is independent of C-holomorphic section at a point of M^{2n-1}. In this section, we have the main result of the present paper that if the Sasakian manifold S^{2n+3} is of constant C-holomorphic sectional curvature, then there exists a submanifold M^{2n-1} whose C-holomorphic sectional curvature is also constant.

§ 1. $(f, g, u, v, w, \lambda, \mu, \nu)$-structure.

Let M^{2n+1} be a differentiable snbmanifold of codimension 3 in a $(2n+4)$-dimensional Euclidean space E^{2n+4}. We denote by P the position vector with initial point at the origin of E^{2n+4} and the terminal point at a point of M^{2n+1} and by C, D and E an orthogonal triple normal frame at a point of M^{2n+1}. If we put $dP=B$ then the Riemannian metric induced on M^{2n+1} is given by

$$g(X, Y)=BX\cdot BY, \tag{1.1}$$

the dot denoting the inner product in E^{2n+4}. We have moreover

$$BX\cdot C=0, \quad BX\cdot D=0, \quad BX\cdot E=0,$$

$$C\cdot C=D\cdot D=E\cdot E=1, \quad C\cdot D=D\cdot E=E\cdot C=0$$

for an arbitrary vector field X of M^{2n+1}.

We denote by J a natural Kaehlerian structure tensor of E^{2n+4}: $J^2=-1$ and by Ω the corresponding 2-form.

The transform JBX of BX by J is written as

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(1. 2) $$JBX=BfX+u(X)C+v(X)D+w(X)E$$

where f is a tensor field of type (1, 1) and u, v and w are 1-forms of M^{2n+1}.

Since $JBX\cdot JBY=BX\cdot BY$, we have from (1. 2),

(1. 3) $$g(fX, fY)=g(X, Y)-u(X)u(Y)-v(X)v(Y)-w(X)w(Y).$$

Since JC is orthogonal to C, JD is orthogonal to D and JE is orthogonal to E and
$$JC\cdot D=\Omega(C, D)=-\Omega(D, C)=-JD\cdot C,\quad JD\cdot E=-JE\cdot D,\quad JE\cdot C=-JC\cdot E,$$
we have equations of the form

(1. 4)
$$JC=-BU+\lambda D-\mu E,$$
$$JD=-BV+\nu E-\lambda C,$$
$$JE=-BW+\mu C-\nu D,$$

where U, V and W are vector fields and λ, μ, ν are functions of M^{2n+1}. Since
$$JBX\cdot C=-BX\cdot JC=u(X),\quad JBX\cdot D=-BX\cdot JD=v(X) \text{ and } JBX\cdot E=-BX\cdot JE=w(X),$$
we have from (1. 2) and (1. 4)

(1. 5) $$u(X)=g(U, X),\qquad v(X)=g(V, X),\qquad w(X)=g(W, X).$$

Applying J to the both sides of (1. 2) and (1. 4) and taking account of $J^2=-1$ and (1. 2), (1. 4) again, we have

(1. 6)
$$f^2X=-X+u(X)U+v(X)V+w(X)W,$$
$$u(fX)=\lambda v(X)-\mu w(X),$$
$$v(fX)=\nu w(X)-\lambda u(X),$$
$$w(fX)=\mu u(X)-\nu v(X),$$
$$fU=-\lambda V+\mu W,\quad fV=-\nu W+\lambda U,\quad fW=-\mu U+\nu V,$$
$$u(U)=1-\lambda^2-\mu^2,\qquad u(V)=\nu\mu,\qquad u(W)=\nu\lambda,$$
$$v(U)=\mu\nu,\qquad v(V)=1-\lambda^2-\nu^2,\qquad v(W)=\mu\lambda,$$
$$w(U)=\lambda\nu,\qquad w(V)=\lambda\mu,\qquad w(W)=1-\mu^2-\nu^2.$$

We call the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure a set of a tensor field f of type (1, 1), a Riemannian metric g, three 1-forms u, v, w and three functions λ, μ, ν satisfying (1. 3) and (1. 6).

When the tensor field

(1. 7) $$S(X, Y)=[f, f](X, Y)+du(X, Y)+dv(X, Y)+dw(X, Y)$$

vanishes, where $[f, f]$ is the Nijenhuis tensor formed with f, the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure is said to be normal.

Now equations of Gauss and Weingarten of M^{2n+1} are

$$(1.8)\quad \begin{aligned} &(\nabla_X B)Y = h(X,Y)C + k(X,Y)D + t(X,Y)E,\\ &\nabla_X C = -BHX + l(X)D - m(X)E,\\ &\nabla_X D = -BKX + n(X)E - l(X)C,\\ &\nabla_X E = -BTX + m(X)C - n(X)D, \end{aligned}$$

X, Y being arbitrary vector fields of M^{2n+1}, where ∇ denotes the operator of covariant differentiation with respect to g, $(\nabla_X B)Y = \nabla_{BX}(BY) - B\nabla_X Y$, $h(X,Y) = h(Y,X)$, $k(X,Y) = k(Y,X)$, $t(X,Y) = t(Y,X)$ and h, k, t are the second fundamental tensors with respect to C, D and E respectively, H, K, T being tensor fields of type $(1,1)$ defined by $g(HX,Y) = h(X,Y)$, $g(KX,Y) = k(X,Y)$, $g(TX,Y) = t(X,Y)$ and l, m, n are the third fundamental tensors.

Now differentiating (1.2) and (1.4) covariantly and taking account of $\nabla J = 0$ and (1.8), we find

$$(1.9)\quad \begin{aligned} S(X,Y) = {} & u(X)[fH - Hf]Y + v(X)[fK - Kf]Y + w(X)[fT - Tf]Y\\ & - u(Y)[fH - Hf]X - v(Y)[fK - Kf]X - w(Y)[fT - Tf]X\\ & + [l(X)v(Y) - l(Y)v(X) - m(X)w(Y) + m(Y)w(X)]U\\ & + [-l(X)u(Y) + l(Y)u(X) + n(X)w(Y) - n(Y)w(X)]V\\ & + [m(X)u(Y) - m(Y)u(X) - n(X)v(Y) + n(Y)v(X)]W. \end{aligned}$$

Thus if H, K and T commute with f and $l = m = n = 0$, then $(f, g, u, v, w, \lambda, \mu, \nu)$-structure is normal.

§ 2. A special metric f-structure with complemented frames.

Let S^{2n+2} be the natural sphere of dimension $2n+2$ in a Euclidean space E^{2n+4} and suppose that M^{2n+1} discussed in the last section is a hypersurface of S^{2n+2}.
(We notice that S^{2n+2} does not posses an almost complex structure unless $n=0$ or $n=2$.)

In this case, we choose the third normal E as $-P$, the initial point of P being at the center of the sphere S^{2n+2}. Then we have

$$(2.1)\qquad t(X,Y) = g(X,Y), \qquad TX = X, \qquad m(X) = n(X) = 0$$

in (1.8).

Now differentiating (1.2) and (1.4) covariantly and taking account of $\nabla J = 0$ and (1.8), we find

$$\begin{aligned} &(\nabla_X f)Y = -h(X,Y)U - k(X,Y)V - g(X,Y)W + u(Y)HX + v(Y)KX + w(Y)X,\\ &(\nabla_X u)Y = -h(X,fY) - \lambda k(X,Y) + \mu g(X,Y) + l(X)v(Y),\\ &(\nabla_X v)Y = -k(X,fY) + \lambda h(X,Y) - \nu g(X,Y) - l(X)u(Y), \end{aligned}$$

$$(2.2)\quad (\nabla_X w)Y=\omega(X,Y)-\mu h(X,Y)+\nu k(X,Y),$$
$$\nabla_X\lambda=-h(X,V)+u(KX),$$
$$\nabla_X\mu=h(X,W)-\nu l(X)-u(X),$$
$$\nabla_X\nu=-k(X,W)+\mu l(X)+v(X),$$

where ω is a 2-form defined by

$$\omega(X,Y)=g(fX,Y).$$

We are interested in case in which $\lambda=\mu=\nu=0$, that is, JC, JD and JE are all tangent to M^{2n+1}. In this case, (1.6) become

$$f^2X=-X+u(X)U+v(X)V+w(X)W,$$
$$u(fX)=0,\quad v(fX)=0,\quad w(fX)=0,$$
$$(2.3)\quad fU=0,\quad fV=0,\quad fW=0,$$
$$u(U)=v(V)=w(W)=1,$$
$$u(V)=u(W)=v(U)=v(W)=w(U)=w(V)=0,$$

and consequently the set (f,g,u,v,w) defines a metric f-structure with complemented frames by virtue of (1.3). In this case (2.2) become

$$(\nabla_X f)Y=-h(X,Y)U-k(X,Y)V-g(X,Y)W+u(Y)HX+v(Y)KX+w(Y)X,$$
$$(\nabla_X u)Y=-h(X,fY)+l(X)v(Y),$$
$$(2.4)\quad (\nabla_X v)Y=-k(X,fY)-l(X)u(Y),$$
$$(\nabla_X w)Y=\omega(X,Y),$$
$$h(X,V)=u(KX),\qquad h(X,W)=u(X),\qquad k(X,W)=v(X).$$

Conversely, suppose that there are given, in an odd-dimensional Riemannian manifold M^{2n+1} with metric tensor g, two symmetric tensor fields H and K of type (1,1), a 1-form l, a unit Killing vector field W, two unit vector fields U and V (U,V,W) forming an orthogonal triple frame such that relations (2.4) hold, where ω, h, k, u, v and w are defined by

$$\omega(X,Y)=g(fX,Y),\quad h(X,Y)=h(Y,X)=g(HX,Y),\quad k(X,Y)=k(Y,X)=g(KX,Y),$$
$$u(X)=g(U,X),\quad v(X)=g(V,X),\quad w(X)=g(W,X).$$

We would like to prove that (f,g,U,V,W) satisfying these conditions define a metric f-structure with complemented frames.

First of all, we notice that, W being a unit Killing vector field, (2.5) $$\omega(X, Y)=g(fX, Y)=g(\nabla_X W, Y)=(\nabla_X w)(Y)$$

is skew-symmetric in X and Y.

From $g(W, W)=1$ and the fourth relation of (2.4), we have $w(\nabla_X W)=0$ that is $w(fX)=0$, from which

(2.6) $$fW=0.$$

From $g(W, U)=0$, the second and third relations of (2.4) and (2.5), we have, by covariant differentiation,

$$g(\nabla_X W, U)+g(W, \nabla_X U)=0, \qquad g(fX, U)+g(W, fHX)=0,$$
$$\omega(X, U)+\omega(W, HX)=0,$$

from which, $\omega(W, HX)$ being zero, we have

(2.7) $$fU=0.$$

Similarly from $g(W, V)=0$, we have

(2.8) $$fV=0.$$

Differentiating (2.6) covariantly, and taking account of the first and fourth relations of (2.4), we find

$$-h(X, W)U-k(X, W)V-g(X, W)W+X+f^2X=0.$$

Since $KW=V$, $HW=U$ we find $h(X, W)=u(X)$, $k(X, W)=v(X)$, and from which, we have

$$f^2X=-X+u(X)U+v(X)V+w(X)W.$$

Thus we have the following

THEOREM 2.1. *Suppose that there are given, in an odd-dimensional Riemannian manifold M^{2n+1} with metric tensor g, two symmetric tensor fields H and K of type* (1, 1), *a unit Killing vector field W, two unit vector fields U and V (U, V, W) forming an orthogonal triple frame such that*

(1) $$\nabla_X W=fX \qquad (\textit{definition of } f),$$

(2) $$(\nabla_X f)Y=-h(X, Y)U-k(X, Y)V-g(X, Y)W+u(Y)HX+v(Y)KX+w(Y)X,$$

(3) $$\nabla_X U=fHX, \qquad \nabla_X V=-fKX,$$

(4) $$KW=V, \qquad HW=U,$$

where h, k, u, v and w are defined by

$$h(X, Y)=h(Y, X)=g(HX, Y), \qquad k(X, Y)=k(Y, X)=g(KX, Y),$$

$$u(X)=g(U,X), \qquad v(X)=g(V,X), \qquad w(X)=g(W,X).$$

Then f, g, U, V, W define a metric f-structure with complemented frames.

§ 3. Submanifold of codimension 2 in a Sasakian manifold.

In this section, we consider a submanifold M^{2n+1} of codimension 2 in a $(2n+3)$-dimensional Sasakian manifold S^{2n+3}. Let (Φ, G, ξ, η) be the Sasakian structure on S^{2n+3}. That is to say, Φ is a tensor field of type $(1,1)$, G is a Riemannian metric on S^{2n+3}, ξ is a vector field and η is a 1-form satisfying

$$\Phi^2=-I+\eta\otimes\xi, \quad \Phi\xi=0, \quad \eta\circ\Phi=0, \quad \eta(\xi)=1, \tag{3.1}$$

$$G(\Phi\tilde{X},\Phi\tilde{Y})+\eta(\tilde{X})\eta(\tilde{Y})=G(\tilde{X},\tilde{Y}),$$

$$\tilde{\nabla}_{\tilde{X}}\xi=\Phi\tilde{X}, \quad (\tilde{\nabla}_{\tilde{X}}\Phi)\tilde{Y}=-G(\tilde{X},\tilde{Y})\xi+\eta(\tilde{Y})\tilde{X}, \tag{3.2}$$

where $\tilde{\nabla}$ is the Riemannian connection of G and $\tilde{X}$ and $\tilde{Y}$ are arbitrary vector fields of S^{2n+3}.

Suppose $\pi: M^{2n+1}\to S^{2n+3}$ is an immersion of an orientable manifold M^{2n+1} in S^{2n+3}. The tensor g defined on M^{2n+1} by

$$g(X,Y)=G(\tilde{B}X,\tilde{B}Y)$$

is a Riemannian metric on M^{2n+1}, where $\tilde{B}$ denotes the differential of the immersion π, and X and Y are arbitrary vector fields of M^{2n+1}.

If C and D are two fields of unit normals defined on M^{2n+1} and ∇ is the Riemannian connection of g, then the Gauss and Weingarten equations of M^{2n+1} can be written as

$$(\nabla_X\tilde{B})Y=h(X,Y)C+k(X,Y)D,$$

$$\nabla_X C=-\tilde{B}HX+l(X)D, \tag{3.3}$$

$$\nabla_X D=-\tilde{B}KX-l(X)C,$$

where $(\nabla_X\tilde{B})Y=\tilde{\nabla}_{\tilde{B}X}(\tilde{B}Y)-\tilde{B}\nabla_X Y$, $h(X,Y)=h(Y,X)$, $k(X,Y)=k(Y,X)$ and h, k are the second fundamental tensors with respect to C and D respectively, H, K being tensor fields of type $(1,1)$ defined by $g(HX,Y)=h(X,Y)$, $g(KX,Y)=k(X,Y)$ and l is the third fundamental tensor.

The transform $\Phi\tilde{B}X$ of $\tilde{B}X$ by Φ is written as

$$\Phi\tilde{B}X=\tilde{B}fX+u(X)C+v(X)D, \tag{3.4}$$

where f is a tensor field of type $(1,1)$ and u, v are 1-forms of M^{2n+1}.

We interested in case in which ΦC and ΦD are both tangent to M^{2n+1}. In this case we have equations of the form

$$\Phi C=-\tilde{B}U, \qquad \Phi D=-\tilde{B}V, \tag{3.5}$$

where U and V are vector fields defined by $g(U,X)=u(X)$, $g(V,X)=v(X)$. We put

$$\xi=\tilde{B}W+\nu C+\kappa D.$$

Applying Φ the above equation and to (3. 4) respectively and taking account of the above equation, (3. 4) and (3. 5), we find $\nu=0$, $\kappa=0$, that is

$$\xi=\tilde{B}W \tag{3.6}$$

and

$$f^2X=-X+u(X)U+v(X)V+w(X)W. \tag{3.7}$$

Differentiating (3. 4) covariantly and taking account of (3. 2), (3. 3) and (3. 4), we have

$$\begin{aligned}(\nabla_X f)Y&=-h(X,Y)U-k(X,Y)V-g(X,Y)W+u(Y)HX+v(Y)KX+w(Y)X,\\ (\nabla_X u)Y&=-h(X,fY)+l(X)v(Y),\\ (\nabla_X v)Y&=-k(X,fY)-l(X)u(Y).\end{aligned} \tag{3.8}$$

Similarly, differentiating (3. 6) and taking account of (3. 3), we find

$$\nabla_X W=fX, \qquad h(X,W)=u(X), \qquad k(X,W)=v(X). \tag{3.9}$$

From $G(\xi,\xi)=1$, we have $g(W,W)=1$ by virtue of (3. 6). Thus we find

$$w(W)=1, \tag{3.10}$$

where w is a 1-form defined by $g(W,X)=w(X)$.

Since C and D are unit vector fields and ΦC and ΦD are both tangent to M^{2n+1}, we find

$$u(U)=1, \qquad v(V)=1 \tag{3.11}$$

by virtue of (3. 1), (3. 5) and (3. 6).

From (3. 6), we have $G(\xi,C)=0$, $G(\xi,D)=0$ and from which we can find that

$$u(V)=0, \qquad v(W)=0, \qquad w(U)=0. \tag{3.12}$$

Thus a set (f,U,V,W,u,v,w) satisfies all conditions of theorem 2. 1. Therefore

$$fU=0, \qquad fV=0, \qquad fW=0 \tag{3.13}$$

and consequently, (f,U,V,W) define a f-structure with complemented frames. Furtheremore from (3. 1) and (3. 4), we can easily verify that

$$g(fX,fY)=g(X,Y)-u(X)u(Y)-v(X)v(Y)-w(X)w(Y). \tag{3.14}$$

Thus we have the following

THEOREM 3. 1. *Let M^{2n+1} be a submanifold of codimension 2 in a Sasakian manifold*

S^{2n+3}, and C, D are two fields of unit normals defined on M^{2n+1}. If ΦC and ΦD are both tangent to M^{2n+1}, where Φ is the Sasakian structure tensor of type $(1,1)$, then M^{2n+1} admits a metric f-structure with complemented frames.

Moreover in our case, differentiating $v(fX)=0$, we obtain $v(\nabla_X f)Y=-(\nabla_X v)fY$. Substituting (3.8) into this equation, we have also

$$HV=KU. \tag{3.15}$$

When the tensor field

$$S(X,Y)=[f,f](X,Y)+du(X,Y)U+dv(X,Y)V+dw(X,Y)W$$

vanishes, where $[f,f]$ is the Nijenhuis tensor formed with f, the (f,g,u,v,w)-structure is said to be normal. If we substitute (3.8) and (3.9) into the above equation, we have

$$\begin{aligned} S(X,Y)=&u(X)(fH-Hf)Y-u(Y)(fH-Hf)X \\ &+v(X)(fK-Kf)Y-v(Y)(fK-Kf)X. \end{aligned} \tag{3.16}$$

Thus if H and K commute with f, then (f,g,u,v,w)-structure is normal.

Conversely, if (f,g,u,v,w)-structure is normal, then $S(X,Y)=0$ for arbitrary vector fields X and Y.

From $S(X,U)=0$ and $S(X,V)=0$, we easily obtain

$$\begin{aligned} fH-Hf&=u(X)fHU+v(X)fKU, \\ fK-Kf&=u(X)fHV+v(X)fKV, \end{aligned} \tag{3.17}$$

and from which, transvecting with u and v respectively, we find

$$u(Hf)=0, \qquad v(Hf)=0, \qquad u(Kf)=0, \qquad v(Kf)=0 \tag{3.18}$$

by virtue of $u\circ f=0$ and $v\circ f=0$.

Since h and k are both symmetric and ω defined by $\omega(X,Y)=g(fX,Y)$ is skewsymmetric, (3.18) is equivalent to

$$fHU=0, \qquad fHV=0, \qquad fKU=0, \qquad fKV=0. \tag{3.19}$$

Therefore H and K commute with f. Thus we have the following

THEOREM 3.2. *In order that the (f,g,u,v,w)-structure induced on a submanifold M^{2n+1} be normal, it is necessary and sufficient that H and K both commute with f.*

§ 4. Invariant hypersurfaces – I.

Let M^{2n} be a $2n$-dimensional hypersurface of M^{2n+1} discussed in § 3 and suppose that $i: M^{2n}\to M^{2n+1}$ is an immersion of orientable manifold M^{2n} in M^{2n+1}. We denote by B the differential of the immersion i.

We assume that the vector fields V and W are tangent to $i(M^{2n})$, the vector field U is the unit normal to M^{2n} and the tangent space to $i(M^{2n})$ is invariant by f. Then we have

$$V = BV', \quad W = BW' \tag{4.1}$$

for two vector fields V' and W' of M^{2n},

$$u(BX') = 0 \tag{4.2}$$

for any vector field X' of M^{2n}, and

$$fBX' = B\phi X'$$

for a tensor field ϕ of M^{2n} and an arbitrary vector field X' of M^{2n}. For simplicity, we call such a hypersurface an invariant hypersurface with respect to V, W and u.

In this case we have seen [3]

$$\begin{aligned} &\phi^2 X' = -X' + v'(X')V' + w'(X')W', \\ &\phi V' = 0, \quad \phi W' = 0, \quad v' \circ \phi = 0, \quad w' \circ \phi = 0, \\ &v'(V') = 1, \quad w'(W') = 1, \quad v'(W') = 0, \quad w'(V') = 0, \end{aligned} \tag{4.3}$$

where we have put

$$v'(X') = v(BX'), \qquad w'(X') = w(BX').$$

Substituting $X = BX'$ into (3.14), we easily see that

$$g'(\phi X', \phi Y') = g'(X', Y') - v'(X')v'(Y') - w'(X')w'(Y'), \tag{4.4}$$

where g' is the induced metric tensor field on M^{2n}, that is,

$$g'(X', Y') = g(BX', BY').$$

We now compute the expression $S(BX', BY')$ for an invariant hypersurface with respect to V, W and u. We have

$$\begin{aligned} S(BX', BY') = B\{&[\phi X', \phi Y'] - \phi[\phi X', Y'] - \phi[X', \phi Y'] + \phi^2[X', Y'] \\ &+ dv'(X', Y')V' + dw'(X', Y')W'\} \end{aligned} \tag{4.5}$$

because of

$$\begin{aligned} du(BX', BY') = 0, \quad & dv(BX', BY') = dv'(X', Y'), \\ & dw(BX', BY') = dw'(X', Y'). \end{aligned}$$

Therefore, if a metric f-structure with complemented frames (f, g, u, v, w) is normal, then the metric ϕ-structure with complemented frames (ϕ, g', v', w') induced on an invariant hypersurface with respect to V, W and u is also normal. Thus we have

THEOREM 4.1. *An invariant hypersurface with respect to V, W and u of a manifold* *with normal metric f-structure with complemented frames* (f, g, u, v, w) *admits an normal metric ϕ-structure with complemented frames* (ϕ, g', v', w').

From the first equation of (3.8) we can easily see that

$$d\omega(X, Y, Z)=0,$$

where $\omega(X, Y)=g(fX, Y)$ and X, Y and Z are arbitrary vector fields of the M^{2n+1}. Using the relation

$$\nabla_{BX'}\omega(BY', BZ')=\nabla_{X'}\varphi(Y', Z'),$$

where $\varphi(X', Y')=g'(\phi X', Y')$ and X', Y' and Z' are arbitrary vector fields of M^{2n}, we have

$$d\varphi(X', Y', Z')=0. \tag{4.6}$$

Next, we consider the sectional curvature with respect to the section determined by X' and W' for any unit vector field X' orthogonal to W' and we denote it by $\rho(X', W')$. Then we have

$$\rho(X', W')=-R'(X', W', X', W'), \tag{4.7}$$

where R' is the curvature tensor field of the hypersurface M^{2n}.

On the other hand, using the first equation of (3.9), we have

$$\nabla_{BX'}W=\nabla'_{X'}(BW'),$$

$$fBX'=t'(X', W')U+B\nabla'_{X'}W',$$

where t' is the second fundamental tensor field for the immersion i and ∇' is the Riemannian connection of g'.

Taking account of (4.3), we can see that $T'W'=0$, where $t'(X', Y')=g'(T'X', Y')$, and

$$\nabla'_{X'}W'=\phi X'. \tag{4.8}$$

From (4.7) and (4.8), we have

$$\rho(X', W')=R'(X', W', W', X')$$
$$=g'(\nabla'_{X'}\nabla'_{W'}W'-\nabla'_{W'}\nabla'_{X'}W'-\nabla'_{[X', W']}W', X').$$

Now

$$\nabla'_{W'}(\phi X')=\nabla'_{\phi X'}W'+[W', \phi X']=\phi^2X'+[W', \phi X'],$$

$$[W', \phi X']+\phi[X', W']=(\mathcal{L}_{W'}\phi)X',$$

where $\mathcal{L}$ is the operator of Lie differentiation. Thus we have

$$\rho(X', W')=g'(-\phi^2X'-(\mathcal{L}_{W'}\phi)X', X'). \tag{4.9}$$

Since $\phi W'=0$ and $\nabla'_{X'}W'=\phi X'$, we have $\varphi=dw'$ and

(4.10) $$\mathcal{L}_{W'}g'=0,$$

Also we have

$$(\mathcal{L}_{W'}\varphi)(X',Y')=(\nabla'_{W'}\varphi)(X',Y')+\varphi(\nabla'_{X'}W',Y')+\varphi(X',\nabla'_{Y'}W')$$
$$=(\nabla'_{W'}\varphi)(X',Y')+(\nabla'_{X'}\varphi)(Y',W')+(\nabla'_{Y'}\varphi)(W',X')$$
$$=d\varphi(W',X',Y')=0$$

by virtue of (4.6).

Thus we have

$$\mathcal{L}_{W'}\phi=0.$$

Therefore from (4.9), we have

(4.11) $$\rho(X',W')=1-v'(X')v'(X').$$

Thus we have the following

THEOREM 4.2. *Let M^{2n} is an invariant hypersurface of M^{2n+1} with respect to V, W and u. Then the sectional curvature with respect to the section determined by two unit vector fields W' and X' is equal to $1-v'(X')v'(X')$, where X' is orthogonal to W'.*

We can prove a similar theorem for the invariant hypersurface with respect to U, W and v.

§ 5. Invariant hypersurfaces - II.

First of all, we recall the equation (4.8) of the last section, that is,

(5.1) $$\nabla'_{X'}W'=\phi X',$$

where ∇' is the Riemannian connection of the induced metric tensor field g' from the metric tensor field g and X' is an arbitrary vector field of an invariant hypersurface M^{2n} of M^{2n+1} with respect to V, W and u.

We put

(5.2) $$HBX'=BH'X'+\tau'(X')U,$$

(5.3) $$KBX'=BK'X'+\mu'(X')U.$$

Now consider the immersion $p=\pi\circ i : M^{2n}\longrightarrow S^{2n+3}$, and denote by B^* the differential of the immersion p. The Gauss equation for this immersion is

(5.4) $$(\nabla'_{X'}B^*)Y'=g'(T'X',Y')\tilde{B}U+g(HBX',BY')C+g(KBX',BY')D$$
$$=g'(T'X',Y')\tilde{B}U+g'(H'X',Y')C+g'(K'X',Y')D,$$

and thus t', h' and k' are the second fundamental tensor fields of p.

On the other hand, substituting (5.2) into the equation $HW=U$ which is equivalent to the second equation of (3.9), we have

$$H'W'=0, \qquad \tau'(W')=1. \tag{5.5}$$

Similarly from the equation $KW=V$, we can easily see that

$$K'W'=V', \qquad \mu'(W')=0. \tag{5.6}$$

Differentiating $fBX'=B\phi X'$ covariantly along $i(M^{2n})$, we can see that

$$(\nabla'_{X'}\phi)Y'=-k'(X',Y')V'-g'(X',Y')W'+v'(Y')K'X'+w'(Y')X', \tag{5.7}$$

$$-h'(X',Y')+\mu'(X')v'(Y')=t'(X',\phi Y'). \tag{5.8}$$

Next, differentiating $V=BV'$ covariantly along $i(M^{2n})$ and using the third equation of (3.8), we can find that

$$(\nabla'_{X'}v')Y'=-k'(X',\phi Y'). \tag{5.9}$$

If the (f, g, u, v, w)-structure induced on M^{2n+1} is normal, then by theorem 3.2, H and K commute with f. The condition that H and f commute: $(fH-Hf)BX'=0$ implies

$$\tau'(\phi X')=0, \qquad \phi H'-H'\phi=0, \tag{5.10}$$

and the condition that K and f commute: $(fK-Kf)BX'=0$ implies

$$\mu'(\phi X')=0, \qquad \phi K'-K'\phi=0. \tag{5.11}$$

Defining a vector field P' in M^{2n} by

$$g'(P',X')=\tau'(X'), \tag{5.12}$$

we have from the first equation of (5.10)

$$\phi P'=0.$$

Since $\phi^2P'=0$, we obtain

$$P'=v'(P')V'+w'(P')W'.$$

On the other hand, since $HW=U$ and $\tau'(W')=1$, we have $w'(P')=1$, and therefore

$$P'=\alpha V'+W', \tag{5.13}$$

where we have put $\alpha=v'(P')$.

Differentiating (5.13) covariantly, we have

$$\nabla'_{X'}P'=\alpha\phi K'X'+\phi X', \tag{5.14}$$

from which we can see that $\nabla'_{X'}\alpha=\nabla'_{X'}(v'(P'))=0$ by virtue of (5.9). Thus α

is a constant.

If $\alpha=0$ then $P'=W'$. Therefore we suppose that $\alpha\neq 0$. In this case, if $\nabla'_{X'}P'=0$, then applying ϕ to (5.14), we find

(5.15) $$\alpha k'(X', Y')+g'(X', Y')$$
$$=\alpha[v'(K'X')v'(Y')+v'(X')w'(Y')]+v'(X')v'(Y')+w'(X')w'(Y'),$$

by virtue of $w'(K'X')=v'(X')$, from which, $k'(X', Y')$ being symmetric, we have

$$v'(K'X')v'(Y')+v'(X')w'(Y')=v'(K'Y')v'(X')+v'(Y')w'(X').$$

Putting $Y'=V'$ in this equation, we find

(5.16) $$k'(X', V')=k'(V', V')v'(X')+w'(X').$$

Substituting (5.16) into (5.15), we find

(5.17) $$\alpha K'X'=-X'+[(1+\alpha\rho)v'(X')+\alpha w'(X')]V'+[\alpha v'(X')+w'(X')]W',$$

where we have put

$$\rho=k'(V', V').$$

Conversely, if K' has the form (5.17), then we have $\alpha\phi K'X'=-\phi X'$, that is, $\nabla'_{X'}P'=0$. Thus we have the following

THEOREM 5.1. *Let M^{2n} be an invariant hypersurface of M^{2n+1} with respect to V, W and u and suppose that the f-structure with complemented frames induced on M^{2n+1} from S^{2n+3} is normal. Then there exists a vector field P' on M^{2n} such that $\phi P'=0$. Moreover, in order that the vector field P' is parallel, it is necessary and sufficient that the second fundamental tensor field k' has the form* (5.17).

In the case in which the vector field P' is parallel, M^{2n} is locally decomposable [1]. Therefore, by the completeness, there is a hypersurface M^{2n-1} of M^{2n} such that $M^{2n}=M^{2n-1}\times M^1$ and P' is normal to M^{2n-1}. Since P' is parallel, we can see that M^{2n-1} is a totally geodesic hypersurface of M^{2n} by the Weingarten equation for the immersion $\pi' : M^{2n-1} \longrightarrow M^{2n}$.

Put $\frac{1}{\sqrt{1+\alpha^2}} P'=Q'$. Then Q' is unit normal vector field to M^{2n-1} and $\phi Q'=0$, and from which it can be seen that there exist a tensor field ϕ^* of type (1, 1) and vector fields V'' and W'' on M^{2n-1} such that

(5.18) $$\phi B'X^*=B'\phi^*X^*,$$

(5.19) $$V'=B'V''+\gamma Q', \qquad W'=B'W''+\delta Q',$$

where B' denotes the differential of the immersion π' and $\gamma=\frac{\alpha}{\sqrt{1+\alpha^2}}$, $\delta=\frac{1}{\sqrt{1+\alpha^2}}$, and X^* is an arbitrary vector field of M^{2n-1}.

Applying ϕ to (5.18), we obtain

$$\phi^{*2}X^* = -X^* + v''(X^*)V'' + w''(X^*)W'',$$

$$\gamma v'' + \delta w'' = 0,$$

where v'' and w'' are 1-forms defined on M^{2n-1} by $g^*(X^*, V'') = v''(X^*)$, $g^*(X^*, W'') = w''(X^*)$ and g^* is the induced metric tensor field on M^{2n-1} by $g'(B'X^*, B'X^*) = g^*(X^*, Y^*)$.

Putting $\frac{1}{\gamma}W'' = W^*$, we have

$$\phi^{*2}X^* = -X^* + w^*(X^*)W^*, \qquad w^*(W^*) = 1. \tag{5.20}$$

Since $v' \circ \phi = 0$ and $w' \circ \phi = 0$, we can easily see that

$$\phi^* W^* = 0, \qquad w^* \circ \phi^* = 0. \tag{5.21}$$

Thus M^{2n-1} admits an almost contact structure (ϕ^*, W^*, w^*). Therefore we have the following

THEOREM 5.2. *Let M^{2n} be an invariant hypersurface of M^{2n+1} with respect to V, W and u, and suppose that the f-structure induced on M^{2n+1} from S^{2n+3} is normal. If the second fundamental tensor field k' has the form* (5.17) *and M^{2n} be a complete orientable hypersurface, then there exists an almost contact submanifold M^{2n-1} such that*

$$M^{2n} = M^{2n-1} \times M^1.$$

We can prove a similar theorem for the invariant hypersurface with respect to U, W and v.

§ 6. Invariant hypersurfaces – III.

In this section, we consider again an invariant hypersurface M^{2n} of a f-manifold with complemented frames M^{2n+1} which is a submanifold of codimension 2 in a Sasakian manifold S^{2n+3}.

If M^{2n} is an invariant hypersurface with respect to V, W and u, then M^{2n} is a submanifold of codimension 3 in S^{2n+3} and from (5.4) $\tilde{B}U$, C and D are unit normals to M^{2n}, and t', h' and k' are the second fundamental tensors with respect to $\tilde{B}U$, C and D respectively.

In this case, if

$$t'(X', Y') = \beta v'(X')v'(Y')$$

for a certain scalar function β, we say that M^{2n} is *v'-cylindrical* with respect to $\tilde{B}U$ [2]. When M^{2n} is cylindrical with respect to $\tilde{B}U$, from (5.8), we see that

$$h'(X', Y') = \beta' v'(X')v'(Y')$$

by virtue of the symmetric property of h', that is, M^{2n} is also v'-cylindrical with respect to C. From (5. 8), we obtain

$$(6.1)\qquad -t'(X', Y')+\nu'(X')v'(Y')=h'(X', \phi Y'),$$

where $\nu'(X')=t'(X', V')$, and from which we see that if M^{2n} is v'-cyclindrical with respect to C, then M^{2n} is also v'-cylindrical with respect to $\tilde{B}U$.

Suppose that M^{2n} is v'-cylindrical with respect to $\tilde{B}U$ or C. Then the equation of Gauss is written by

$$(6.2)\qquad \tilde{R}(B^*X', B^*Y', B^*Z', B^*A')$$
$$=R'(X', Y', Z', A')-k'(Y', Z')k'(X', A')-k'(X', Z')k'(Y', A'),$$

by virtue of (5. 4), where $\tilde{R}$ and R' are the curvatue tensor fields of S^{2n+3} and M^{2n} respectively and X', Y', Z' and A' are arbitrary vector fields of M^{2n}.

If S^{2n+3} is of constant C-holomorphic sectional curvature, then

$$\begin{aligned}\tilde{R}(\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{A}) = (c'+1)[&G(\tilde{Y}, \tilde{Z})G(\tilde{X}, \tilde{A})-G(\tilde{Y}, \tilde{A})G(\tilde{X}, \tilde{Z})]\\ -c'[&\eta(\tilde{X})\{\eta(\tilde{A})G(\tilde{Y}, \tilde{Z})-\eta(\tilde{Z})G(\tilde{Y}, \tilde{A})\}\\ &-\eta(\tilde{Y})\{\eta(\tilde{A})G(\tilde{X}, \tilde{Z})\}-\eta(\tilde{Z})G(\tilde{X}, \tilde{A})\}\\ &-G(\Phi\tilde{X}, \tilde{A})G(\Phi\tilde{Y}, \tilde{Z})+G(\Phi\tilde{Y}, \tilde{A})G(\Phi\tilde{X}, \tilde{Z})+2G(\Phi\tilde{X}, \tilde{Y})G(\Phi\tilde{Z}, \tilde{A})],\end{aligned}$$

where c' ia a constant and $\tilde{X}, \tilde{Y}, \tilde{Z}$ and $\tilde{A}$ are arbitrary vector fields of S^{2n+3}.

In this case, setting $\tilde{X}=B^*X'$, $\tilde{Y}=B^*Y'$, $\tilde{Z}=B^*Z'$ and $\tilde{A}=B^*A'$, we obtain

$$(6.3)\qquad \begin{aligned}R'(X', Y', Z', A')=&k'(Y', Z')k'(X', A')-k'(X', Z')k'(Y', A')\\ &+(c'+1)[g'(Y', Z')g'(X', A')-g'(X', Z')g'(Y', A')]\\ &-c'[w'(X')\{w'(A')g'(Y', Z')-w'(Z')g'(Y', A')\}\\ &-w'(Y')\{w'(A')g'(X', Z')-w'(Z')g'(X', A')\}\\ &-g'(\phi X', A')g'(\phi Y', Z')+g'(\phi Y', A')g'(\phi X', Z')\\ &+2g'(\phi X', Y')g'(\phi Z', A')],\end{aligned}$$

by virtue of (6. 2).

§ 7. Closely related submanifold to a Sasakian manifold.

In this section, we assume that the vector field P' of M^{2n} defined by (5. 12) is parallel, then there exists an invariant hypersurface M^{2n-1} of M^{2n} such that $M^{2n}=M^{2n-1}\times M^1$. We investigate such an invariant hypersurface M^{2n-1} of M^{2n}.

First of all, substituting $X'=B'X^*$ into (4. 4), and taking account of (5. 18) and (5. 19), we have

$$g^*(\phi^*X^*, \phi^*Y^*)=g^*(X^*, Y^*)-\frac{1}{\gamma^2}w''(X^*)w''(Y^*),$$

that is,

$$g^*(\phi^*X^*, \phi^*Y^*) = g^*(X^*, Y^*) - w^*(X^*)w^*(Y^*). \tag{7.1}$$

Thus M^{2n-1} admits an almost contact metric structure (ϕ^*, g^*, W^*, w^*).

In the case in which the ϕ-manifold with complemented frames M^{2n} is normal, computing the expression

$$[\phi B'X^*, \phi B'Y^*] - \phi[\phi B'X^*, B'Y^*] - \phi[B'X^*, \phi B'Y^*] + \phi^2[B'X^*, B'Y^*]$$
$$+ dv'(B'X^*, B'Y^*)V' + dw'(B'X^*, B'Y^*)W' = 0,$$

we can obtain

$$[\phi^*X^*, \phi^*Y^*] - \phi^*[\phi^*X^*, Y^*] - \phi^*[X^*, \phi^*Y^*] + \phi^{*2}[X^*, Y^*] + dw^*(X^*, Y^*)W^* = 0, \tag{7.2}$$

because of

$$\gamma dv''(X^*, Y^*) + \delta dw''(X^*, Y^*) = 0.$$

Differentiating the second equation of (5.19) covariantly along $\pi'(M^{2n-1})$, we find $\nabla^*_{X^*}W'' = \phi^*X^*$, that is,

$$\nabla^*_{X^*}W^* = \frac{1}{\gamma}\phi^*X^*, \tag{7.3}$$

where ∇^* is the Riemannian connection of the metric tensor field g^*.

We can put

$$K'B'X^* = B'K^*X^* + \omega^*(X^*)Q'. \tag{7.4}$$

Next, differentiating (4.3) covariantly along $\pi'(M^{2n-1})$ and taking account of (5.7), (5.19) and (7.4), we obtain

$$\nabla^*_{X^*}\phi^*Y^* = -k^*(X^*, Y^*)V'' - g^*(X^*, Y^*)W'' + v''(Y^*)K^*X^* + w''(Y^*)X^*, \tag{7.5}$$

$$\gamma k^*(X^*, Y^*) + \delta g^*(X^*, Y^*) = \omega^*(X^*)v''(Y^*), \tag{7.6}$$

where $k^*(X^*, Y^*) = g^*(K^*X^*, Y^*)$.

Taking account of the symmetry property of the tensor field k^*, we have from (7.6) that $\omega^*(X^*) = \lambda^*v''(X^*)$, where λ^* is a certain scalar function. Then (7.6) can be written as

$$\gamma k^*(X^*, Y^*) + \delta g^*(X^*, Y^*) = \lambda^*v''(X^*)v''(Y^*). \tag{7.7}$$

Substituting (7.7) into (7.5), we obtain

$$\Delta^*_{X^*}\phi^*Y^* = \frac{1}{\gamma^2}(-g^*(X^*, Y^*)W'' + w''(Y^*)X^*),$$

that is,

$$\nabla^*_{X^*}\phi^* Y^* = \frac{1}{\gamma}(-g^*(X^*, Y^*)W^* + w^*(Y^*)X^*). \tag{7.8}$$

From (5.20), (5.21), (7.1), (7.2), (7.3) and (7.8), we obtain the following

THEOREM 7.1. *Suppose that the vector field P' defined by* (5.21) *in a normal metric ϕ-manifold M^{2n} with complemented frames is parallel. Then the hypersurface M^{2n-1} of M^{2n} such that $M^{2n}=M^{2n-1}\times M^1$ admits a tensor field ϕ^* of type* (1, 1), *a metric tensor field g^*, a vector field W^* and a 1-form w^* such that*

$$\phi^{*2} = -I + w^* \otimes W^*, \quad \phi^* W^* = 0, \quad w^* \circ \phi^* = 0, \quad w^*(W^*) = 1,$$

$$g^*(\phi^* X^*, \phi^* Y^*) + w(^*X^*)w^*(Y^*) = g^*(X^*, Y^*),$$

$$\nabla^*_{X^*} W^* = \frac{1}{\gamma}\phi^* X^*, \tag{7.9}$$

$$(\nabla^*_{X^*}\phi^*)Y^* = \frac{1}{\gamma}(-g^*(X^*, Y^*)W^* + w^*(Y^*)X^*),$$

$$[\phi^*, \phi^*] + dw^* \otimes W^* = 0,$$

where γ is a constant, $\gamma \neq 1$, $[\phi^*, \phi^*]$ is the Nijenhuis tensor field formed with ϕ^* and X^* is an arbitrary vector field of M^{2n-1}.

The set (ϕ^*, g^*, W^*, w^*) satisfying (7.9) seems *closely related to a Sasakian structure.* From the third equation of (7.9), we have

$$\gamma^2(\nabla^*_{Z^*}\nabla^*_{Y^*}w^*)X^* = w^*(Y^*)g^*(X^*, Z^*) - w^*(X^*)g^*(Y^*, Z^*),$$

and using the Ricci formula, we obtain

$$\gamma^2 R^*(Z^*, Y^*, X^*, W^*) = w^*(Z^*)g^*(X^*, Y^*) - w^*(Y^*)g^*(X^*, Z^*), \tag{7.10}$$

where R^* denotes the curvature tensor field of M^{2n-1}.

Similarly, from the fourth equation of (7.9) and the Ricci formula, we obtain

$$\begin{aligned} &\gamma^2 R^*(\phi^* Z^*, \phi^* Y^*, X^*, A^*) \\ &= \gamma^2 R^*(Z^*, Y^*, X^*, A^*) + \varphi^*(Z^*, A^*)\varphi^*(Y^*, X^*) - \varphi^*(Y^*, A^*)\varphi^*(Z^*, X^*) \\ &\quad - g^*(Z^*, A^*)g^*(Y^*, X^*) + g^*(Y^*, A^*)g^*(Z^*, X^*), \end{aligned} \tag{7.11}$$

where we have put

$$\varphi^*(X^*, Y^*) = g^*(\phi^* X^*, Y^*).$$

Moreover we have

$$R^*(\phi^* X^*, Y^*, \phi^* Z^*, A^*) = R^*(\phi^* Z^*, A^*, \phi^* X^*, Y^*). \tag{7.12}$$

Replacing Z^* and X^* by ϕ^*Z^* and ϕ^*X^* respectively in (7.11), we have

$$\begin{aligned}(7.13)\quad &\gamma^2R^*(\phi^*Z^*, Y^*, \phi^*X^*, A^*)-\gamma^2R^*(\phi^*Z^*, Y^*, \phi^*A^*, X^*)\\ &=g^*(Y^*, X^*)g^*(Z^*, A^*)-g^*(Z^*, X^*)g^*(Y^*, A^*)\\ &+w^*(Z^*)w^*(X^*)g^*(Y^*, A^*)-w^*(Z^*)w^*(A^*)g^*(Y^*, X^*)\\ &-\varphi^*(Y^*, X^*)\varphi^*(Z^*, A^*)+\varphi^*(Z^*, X^*)\varphi^*(Y^*, A^*).\end{aligned}$$

Using the Bianchi identity and taking account of (7.11), we obtain

$$\begin{aligned}(7.14)\quad &\gamma^2R^*(\phi^*Z^*, A^*, \phi^*Y^*, X^*)-\gamma^2R^*(\phi^*Y^*, A^*, \phi^*Z^*, X^*)\\ &=\gamma^2R^*(Z^*, Y^*, A^*, X^*)+\varphi^*(Z^*, X^*)\varphi^*(Y^*, A^*)-\varphi^*(Y^*, X^*)\varphi^*(Z^*, A^*)\\ &\qquad -g^*(Z^*, X^*)g^*(Y^*, A^*)+g^*(Y^*, X^*)g^*(Z^*, A^*).\end{aligned}$$

Now let V^* be any unit vector field on M^{2n-1} and consider the section determined by ϕ^*V^* and $\phi^{*2}V^*$, which are orthogonal to each other. We call this a C-holomorphic section, and the sectional curvature $4k_1$ determined by such a section the C-holomorphic sectional curvature:

$$(7.15)\qquad 4k_1=-\frac{R^*(\phi^*V^*, \phi^{*2}V^*, \phi^*V^*, \phi^{*2}V^*)}{g^*(\phi^*V^*, \phi^*V^*)g^*(\phi^{*2}V^*, \phi^{*2}V^*)}.$$

Taking account of (7.9) and (7.10), (7.15) is written by

$$\begin{aligned}(7.16)\quad &R^*(\phi^*V^*, V^*, \phi^*V^*, V^*)+\frac{1}{\gamma^2}w^*(V^*)w^*(V^*)\{1-w^*(V^*)w^*(V^*)\}\\ &+4k_1\{1-w^*(V^*)w^*(V^*)\}\{1-w^*(V^*)w^*(V^*)\}=0.\end{aligned}$$

Now, if we assume k_1 is independent of the choice of C-holomorphic section at an arbitrary point of M^{2n-1}, then (7.16) should be satisfied for arbitrary V^*. In this case, by the well known method (e.g., §2 of [4]) and by some complicated computations, we have, by virtue of (7.10)–(7.14),

$$\begin{aligned}(7.17)\quad R^*(X^*, Y^*, Z^*, A^*)=&\left(c+\frac{1}{\gamma^2}\right)[g^*(Y^*, Z^*)g^*(X^*, A^*)-g^*(Y^*, A^*)g^*(X^*, Z^*)]\\ &-c[w^*(X^*)\{w^*(A^*)g^*(Y^*, Z^*)-w^*(Z^*)g^*(Y^*, A^*)\}\\ &-w^*(Y^*)\{w^*(A^*)g^*(X^*, Z^*)-w^*(Z^*)g^*(X^*, A^*)\}\\ &-\varphi^*(X^*, A^*)\varphi^*(Y^*, Z^*)+\varphi^*(Y^*, A^*)\varphi^*(X^*, Z^*)\\ &+2\varphi^*(X^*, Y^*)\varphi^*(Z^*, A^*)],\end{aligned}$$

where c is a constant defined by $4c=4k_1-\frac{1}{\gamma^2}$. Thus we have the following

THEOREM 7.2. *If, in a manifold M^{2n-1} with a structure (ϕ^*, g^*, W^*, w^*) satisfying* (7.9), *the C-holomorphic sectional curvature is independent of C-holomorphic section at a point of M^{2n-1}, then the curvature tensor of M^{2n-1} has components of the form* (7.17), *where c and γ are constants and $n\neq 1$ and $\gamma\neq 1$.*

Next, we assume that all hypotheses of theorem 5.2 are satisfied. In this case we can consider a submanifold M^{2n-1} with a structure (ϕ^*, g^*, W^*, w^*) satisfying (7.9) such that $M^{2n}=M^{2n-1}\times M^1$, where M^{2n} is an f-manifold with complemented frames.
Moreover we assume that above M^{2n} is a v'-cylindrical submanifold of codimension 3 in a Sasakian manifold S^{2n+3} with respect to $\tilde{B}U$ or C.

Under these assumptions, first of all, we can see easily that

$$d\varphi^*(X^*, Y^*, Z^*)=0. \tag{7.18}$$

Using the relation

$$\nabla^*_{X^*}W''=\phi^*X^*\quad .$$

and computing the sectional curvature with respect to the section determined by X^* and W^* for any unit vector field X^* orthogonal to W'':

$$\begin{aligned}&R^*(X^*, W'', W'', X^*)\\&=g^*(\nabla^*_{X^*}\nabla^*_{W''}W''-\nabla^*_{W''}\nabla^*_{X^*}W''-\nabla^*_{[X^*, W'']}W'', X^*)\end{aligned}$$

we have, by the similar method used in §4,

$$R^*(X^*, W'', W'', X^*)=g^*(-\phi^{*2}X^*-(\mathcal{L}_{W''}\phi^*)X^*, X^*).$$

Since $\mathcal{L}_{W''}g^*=0$ and

$$\mathcal{L}_{W''}\varphi^*(X^*, Y^*)=d\varphi^*(W'', X^*, Y^*)=0$$

by virtue of (7.18), we have

$$\begin{aligned}R^*(X^*, W'', W'', X^*)&=g^*(X^*-w^*(X^*)W^*, X^*)\\&=g^*(X^*, X^*)-\frac{1}{\gamma^2}w''(X^*)g^*(W'', X^*).\end{aligned} \tag{7.19}$$

On the other hand, , since M^{2n-1} is a totally geodesic hypersurface of M^{2n}, the equation of Gauss is written as

$$R'(B'X^*, B'Y^*, B'Z^*, B'A^*)=R^*(X^*, Y^*, Z^*, A^*). \tag{7.20}$$

If S^{2n+3} is of constant C-holomorphic sectional curvature, then we have the relation (6.3). In this case, setting $X'=B'X^*$, $Y'=B'Y^*$, $Z'=B'Z^*$ and $A'=B'A^*$ in (6.3), taking account of (7.20) and substituting (7.4) into it, we have, after some computations,

$$R^*(X^*, Y^*, Z^*, A^*) = k^*(Y^*, Z^*)k^*(X^*, A^*) - k^*(Y^*, A^*)k^*(X^*, Z^*)$$
$$+(c'+1)[g^*(Y^*, Z^*)g^*(X^*, A^*) - g^*(Y^*, A^*)g^*(X^*, Z^*)]$$
$$(7.21) \qquad -c'[w''(X^*)\{w''(A^*)g^*(Y^*, Z^*) - w''(Z^*)g^*(Y^*, A^*)\}$$
$$-w''(Y^*)\{w''(A^*)g^*(X^*, Z^*) - w''(Z^*)g^*(X^*, A^*)\}$$
$$-\varphi^*(X^*, A^*)\varphi^*(Y^*, Z^*) + \varphi^*(Y^*, A^*)\varphi^*(X^*, Z^*)$$
$$+2\varphi^*(X^*, Y^*)\varphi^*(Z^*, A^*)].$$

For the case in which the unit vector field X^* is orthogonal to W'', from (7.21), we have

$$(7.22) \qquad R^*(X^*, W'', W'', X^*) = k^*(W'', W'')k^*(X^*, X^*) - k^*(X^*, W'')k^*(X^*, W'') + (c'+1)\gamma^2 - c'\gamma^4,$$

by virtue of

$$g^*(W'', W'') = g'(B'W'', B'W'') = g'(W' - \delta Q', \ W' - \delta Q') = 1 - \delta^2 = \gamma^2.$$

On the other hand, from (7.7), since X^* is a unit vector field orthogonal to W'', we have

$$k^*(X^*, X^*) = -\frac{\delta}{\gamma}, \qquad k^*(X^*, W'') = 0, \qquad k^*(W'', W'') = \gamma\delta(-1 + \lambda^*\delta).$$

Substituting these relations into (7.22) and comparing the result with (7.19), we have

$$-\delta^2(-1 + \lambda^*\delta) + (c'+1)\gamma^2 - c'\gamma^4 = 1,$$

that is,

$$(7.23) \qquad \lambda^* = \frac{\gamma^2}{\delta}c'.$$

Thus, (7.7) is written as

$$(7.7)' \qquad k^*(X^*, Y^*) = -\frac{\delta}{\gamma}g^*(X^*, Y^*) + \frac{\gamma}{\delta}c'v''(X^*)v''(X^*)$$
$$= -\frac{\delta}{\gamma}g^*(X^*, Y^*) + \gamma\delta c' w^*(X^*)w^*(X^*).$$

Substituting (7.7)′ into (7.21), we obtain

$$R^*(X^*, Y^*, Z^*, A^*) = (c' + \frac{1}{\gamma^2})[g^*(Y^*, Z^*)g^*(X^*, A^*) - g^*(Y^*, A^*)g^*(X^*, Z^*)]$$
$$-c'[w^*(X^*)\{w^*(A^*)g^*(Y^*, Z^*) - w^*(Z^*)g^*(Y^*, A^*)\}$$
$$-w^*(Y^*)\{w^*(A^*)g^*(X^*, Z^*) - w^*(Z^*)g^*(X^*, A^*)\}$$

$$-\varphi^*(X^*, A^*)\varphi^*(Y^*, Z^*)+\varphi^*(Y^*, A^*)\varphi^*(X^*, Z^*)$$

$$+2\varphi^*(X^*, Y^*)\varphi^*(Z^*, A^*)].$$

Therefore, by the theorem 7.2, we conclude that M^{2n-1} is of constant C-holomorphic sectional curvature. Thus we have the following

THEOREM 7.3. *We assume that all hypotheses of theorem* 5.2 *are satisfied and let* M^{2n} *be a* v'*-cylindrical submanifold of* S^{2n+3} *with respect to one of three normal vector fields defined on* M^{2n}. *In this case, if the Sasakian manifold* S^{2n+3} *is of constant C-holomorphic sectional curvature, then the submanifold* M^{2n-1} *with the structure* (ϕ^*, g^*, W^*, w^*) *satisfying* (7.9) *is also of constant C-holomorphic sectional curvature.*

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