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ON THE VECTOR VALUED MEASURES

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The first striking theorem on the range of a vector valued measure was Liapounoff's theorem [5] appeared in 1940 which says that the range of a measure with values in a finite dimensional vector space is compact and when the measure is atomfree the range is also convex. Then in 1948 Halmos [4] somewhat simplified the proof of the Liapounoff's theorem. In 1966 Lindenstrauss [6] shortened the proof of the theorem drastically. Olech [7] in 1968 investigated the range of unbounded vector valued measure, on the same year Rieffel [8] generalized the Radon-Nikodym theorem to vector valued measures employing the Bochner integral. In 1969 Uhl [10] showed that a vector valued measure with bounded variation whose values are either in a reflexive space or in a separable dual space has a precompact range, moreover, if the measure is atomfree the range is convex. Finally, in 1973 A. Tong and the author [1] extended Rieffel's Radon-Nikodym theorem and Uhl's result on the range of a Banach space valued measure. We generalize in this note the result on the range of a vector valued measure in [1] to the measure with its values in a Fréchet space.

Let Σ be a σ-algebra of sets. By a vector valued measure we mean a countably additive set function defined on Σ whose values in a topological vector space.

We begin with the Liapounoff's theorem.

THEOREM (Liapounoff). *Let $\mu_1, \mu_2, \cdots, \mu_n$ be real-valued atomfree measures on a σ-algebra Σ. Then the set of points in R^n of the form $(\mu_1(E), \mu_2(E), \cdots, \mu_n(E))$ where $E\in\Sigma$ is compact and convex.* [5], [9].

In the proof of this theorem the Radon-Nikodym theorem does an important role. In fact if we define

$$\mu=|\mu_1|+|\mu_2|+\cdots+|\mu_n|$$

where $|\mu_i|$ is the total variation of μ_i, then each μ_i is absolutely continuous with respect to μ and there exist functions f_i in $L^1(\mu)$ such that $d\mu_i=f_i d\mu$. This fact enables us to show that the linear operator $T: L^\infty(\mu)\to R^n$ defined by

$$T(g)=\left[\int gf_1d\mu, \quad \int gf_2d\mu, \cdots\cdots \int gf_nd\mu\right]$$

for each bounded μ-measurable real valued functions, is weak* continuous. For the detailed proof the reader may consult [6] or [9].

When $\mu:\Sigma\to B$ is a vector valued measure where B is either a reflexive Banach space or a separable dual space Dunford-Pettis theorem [3] guarantees the existance of a B-valued $|\mu|$-integrable function f such that $\mu(E)=\int_E fd\mu$ for all E in Σ. Uhl utilized this

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theorem to show the compactness of the closure of the range of μ.

THEOREM (Uhl). *Let B be a Banach space which is either reflexive or separable dual. If $\mu : \Sigma \to B$ is a measure of bounded variation, then the range of μ is precompact in B. Morever if μ is atomfree, the closure of the range of μ is compact convex.* [10].

Although the conditions for his theorem seem to be rather complicated Rieffel is believed to be the first mathematician to generalize Radon-Nikodym theorem for Bochner integral to a general Banach space.

THEOREM (Rieffel). *Let (X, Σ, μ) be a σ-finite positive measure space and let B be a Banach space. Let m be a B-valued measure on Σ. Then m is the indefinite integral with respect to μ of a B-valued Bochner integrable function on X if and only if*

(1) *$m(E)=0$ whenever $\mu(E)=0$, $E \in \Sigma$*

(2) *the total variation, $|m|$, of m is a finite measure,*

(3) *given $E \in \Sigma$ with $0 < \mu(E) < \infty$ there is an $F \in E$ such that $\mu(F) > 0$ and*

$$A_F(m) = \{m(F')/\mu(F') : F' \subset F, \mu(F') > 0\}$$

is precompact. [8].

The condition (3) has been slightly improved in the following theorem.

THEOREM (Cho-Tong) A. *Let (X, Σ, μ) be a σ-finite measure space. Let $m : \Sigma \to B$ be a B-valued measure where B is a Banach space. Then m is the indefinite integral with respect to μ of a Bochner integrable function $f : X \to B$ if and only if*

(1) *$m(M)=0$ whenever $\mu(M)=0$, $M \in \Sigma$*

(2) *m has a finite total variation*

(3) *given $M \in \Sigma$ with $0 < \mu(M) < \infty$, there is a set $N \in \Sigma$ such that $\mu(N) > 0$, $N \subset M$ and N satisfies the following condition: if $\{N_i\}$ is any sequence of disjoint (non null) measurable sets in N, then $\{m(N_i)/\mu(N_i) : i=1, 2, \cdots\}$ is a precompact set.* [1].

Also the following theorem slightly generalizes the Uhl's theorem.

THEOREM (Cho-Tong) B. *Let (X, Σ, μ) be a finite measure space and let $m : \Sigma \to B$ a B-valued measure where B is a Banach space. If the set*

$$\{m(M_i)/\mu(M_i) : \mu(M_i) > 0 \text{ and } M_i \in \Sigma\}$$

is precompact for each sequence $\{M_i\}$ of disjoint measurable sets, then the range of m is precompact. [1].

We will generalize in the following the above Theorem B to an F-valued measure where F is a Fréchet space.

LEMMA 1. *Let (X, Σ, μ) be an atomfree positive measure space and let $T : L_1(\mu) \to F$ be a continuous linear operator where F is a Fréchet space. For each positive real number α difine $R(\alpha) = \{\chi_M/\mu(M)\ M \in \Sigma, 0 < \mu(M) < \alpha\}$ where χ_M is the characteristic function* *of M. Then*

(1) *for all α, β with $0<\alpha<\beta$, $R(\beta)$ is a subset of convex hull of $R(\alpha)$,*

(2) *$R(\beta)$ is a precompact set if and only if there is a positive real number α less then β such that $R(\alpha)$ is a precompact set.*

Proof. (1) Let M be a measurable set with $0<\mu(M)<\beta$. There is a disjoint decomposition $\{M_1, M_2, \cdots, M_n\}$ of M where $M_i \in \Sigma$ and $0<\mu(M_i)<\alpha$, $i=1, 2, \ , n$. Hence, $\mu(M) = \Sigma_{i=1}^n \mu(M_i)$ and $\Sigma_{i=1}^n \mu(M_i)/\mu(M)=1$. Now

$$T(\chi_M/\mu(M)) = \Sigma_{i=1}^n \ (\mu(M_i)/\mu(M))T(\chi_M/\mu(M_i))$$

is a member of the convex hull of $R(\alpha)$.

(2) In a Fréchet space the closed convex hull of a compact set is compact[9]. Therefore, if $R(\alpha)$ is precompact, then its convex hull is also precompact and by (1) $R(\beta)$ should be a precompact set for all β with $0<\alpha<\beta$.

LEMMA 2. *Let $A_1 \supset A_2 \supset \cdots$ be a sequence of nonprecompact bounded sets in a Fréchet space X such that the convex hull of A_{n+1} contains A_n, $n=1, 2, \cdots\cdots\cdots$. Then there exists a fixed positive constant ε such that none of A_i is covered by a finite number of ε-balls.*

Proof. Since each A_i is not precompact but bounded there exists a sequence $\{\varepsilon_n\}$ of positive numbes such that A_n can not be covered by a finite number of ε_n-balls wherease it can be covered by a finite number of $(2\varepsilon_n)$-balls. Suppose that ε_n converges to zero as n tends to ∞ and let δ be an arbitrary positive number. Without loss of generality we may assume $\varepsilon_{n+1}<\varepsilon_n$. Choose a convex neighborhood V of 0 in X such that $V+V \subset B(0, \varepsilon)$, where $B(0, \delta)$ is the δ-ball with the center at O, then choose a sufficiently large n such that $B(0, 2\varepsilon_n) \subset V$. By the choice of ε_n there exists a finite set $E=\{e_1, e_2, \cdots\cdots, e_m\}$ such that $A_n \subset E+V$. Let E_1 be the convex hull of E, then E_1 is compact. Let $x \in A_1$. Since the cnvex hull of A_n, contains A_1 x can be written as $x=\sum_{i=1}^k t_i x_i$ where $x_i \in A_n$, $t_i \geq 0$, $i=1, 2, 3, \cdots, k$, and $\sum_{i=1}^k t_i=1$. For each i, there is an element y_i of E such that $x_i - y_i \in V$. Writing

$$x=\sum_{i=1}^k t_i y_i + \sum_{i=1}^k t_i (x_i - y_i)$$

we see that $x \in E_1 + V$. Therefore $A_1 \subset E_1 + V$. But E_1 is compact and there is a finite set F such that $E_1 \subset F+V$, and hence we have

$$A_1 \subset E_1 + V \subset F + V + V \subset F + B(0, \delta)$$

and A_1 is totally bounded which contradicts the hypothesis. Therefore ε_n does not converge to zero as n tends to ∞ and by the same argument ε_n has no subsequence converging to zero and hence there exists the required constant $\varepsilon>0$.

THEOREM 1. *Let (X, Σ, μ) be an atomfree positive measure space and let F be a Fréchet space. Then a bounded linear operator $T: L^1(\mu) \to F$ is compact if and only if the set*

$$\{T(\chi_{M_i}/\mu(M_i) : M_i \in \Sigma, \ \mu(M_i)>0\}$$

is precompact for every sequence of disjoint measurable sets $\{M_i\}$.

Proof. Since the convex hull of $R(\alpha)$ contains the union of all of the $R(\beta)$ for all $\beta > \alpha$ by lemma 1 and simple functions are dense in the unit sphere of $L^1(\mu)$ it is enough to show that there is a positive real number α such that $R(\alpha)$ is precompact.

Suppose the contrary, then none of $R\left(\frac{1}{n}\right)$, $n=1, 2, \cdots$, is precompact and by lemma 2 there is a constant $\varepsilon>0$ such that none of $R\left(\frac{1}{n}\right)$ can be covered by a finite number of ε-balls $B(y_i, \varepsilon)=\{y\in F : d(y_i, y)<\varepsilon\}$. Let $y_1\in R(1)$. By induction choose a sequence $\{y_i\}_{i=1}^{\infty}$ such that $y_n\in R\left(\frac{1}{n}\right)\sim\cup_{i=1}^{n-1}B(y_i, \varepsilon)$. Each y_i is apart at least the distance of ε and the sequence has no convergent subsequence. Since $y_n\in R\left(\frac{1}{n}\right)$ there is a measurable set M_n such that

$$y_n=T(\chi_{M_n}/\mu(M_n)),\quad n=1, 2, \cdots\cdots$$

and

$$\mu(M_n)<\frac{1}{n},\quad n=1, 2, \cdots\cdots$$

Choose a subsequence $\{\alpha_i\}$ of $\{\mu(M_n)\}$ such that

$$\alpha_{i+1}<2^{-i}\alpha_i$$

Let $\alpha_i=\mu(M_{n(i)})$, and difine a sequence $\{N_i\}$ of disjoint measurable sets by

$$N_i=M_{n(i)}-\cup_{j>i}M_{n(j)}$$

Then

$$\begin{aligned}&\|\chi_{N_i}/\mu(N_i)-\chi_{M_{n(i)}}/\mu(M_{n(i)})\|\\&=1-\mu(N_i)/\mu(M_{n(i)}+\mu(\cup_{j>i}M_{n(j)})/\mu(M_{n(i)})\\&\leqq 3/2^i.\end{aligned}$$

Therefore,

$$T(\chi_{N_i}/\mu(N_i))-T(\chi_{M_{n(i)}}/\mu(M_{n(i)})\to 0$$

as $i\to\infty$ and the sequence $T(\chi_{N_i}/\mu(_{N_i}))$ has no convergent subsequence which contradicts the hypothesis.

COROLLARY 2. *Let* (X, Σ, μ) *be an atomfree positive measure space and let* F *be a Fréchet space. Then a measure* $m:\Sigma\to F$ *has a precompact range if the set* $\{m(N_i)/\mu(N_i) : N_i\in\Sigma,\ \mu(N_i)>0\}$ *is precompact for every* $\{N_i\}$ *of disjoint measurable sets.*

Proof. Let the operator $T: L^1(\mu)\to F$ be a linear extension of m such that $T(\alpha\chi_M+\beta\chi_N)=\alpha m(M)+\beta m(N)$ for characteristic functions χ_M, $M\in\Sigma$. Then T is compact and hence the range of m is precompact.

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