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T-RELATED FRECHET SPACES

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1. Introduction. B. Malgrange introduced in [1] $P(D)$-related Frechet spaces to prove $P(D)\mathcal{D}'^F(\Omega)=\mathcal{D}'^F(\Omega)$ where $P(D)$ is a differential polynomial and $\mathcal{D}'^F(\Omega)$ is the space of distributions in Ω of finite order. The functional analytic version of $P(D)$-related Frechet spaces and its generalizations are developped in F. Treves [2]. The central result in [2] is the corollary 2 of theorem 17.2; it reads in our notations (cf. 2), if the map T from E into F is such that the image of ${}^tT$ is weakly closed in E' and ${}^tT$ is injective, then T from G into M is an epimorphism and has a homogeneous approximation property iff L and M are T-related. In this paper we weaken the condition for T and assuming the condition for T to be presurjective we get the similar results. The methods used here clarifies the connections between presurjectivity of T and T-related Frechet spaces.

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2. Notations and Definitions. Let E and F be locally convex topological linear spaces. We further assume that F is barelled. Let Σ be a locally convex Hausdorff topological space. Let L and M be Frechet spaces. We assume that

i) $E(F)$ is continuously imbedded in L (M resp.) and dense in L (M resp.), and

ii) L and M are continuously imbedded in Σ.

Let T be a continuous transformation from Σ into Σ such that T restricted on E (which we shall denote by T again) is a continuous transformation from E into F. We shall denote by E' the continuous dual of E and by ${}^tT$ the continuous transpose of T as usual. For any $f\in E'$, $|f|(x)=|f(x)|$ for every $x\in E$. For any seminorm q on F, ${}^tTq$ is a seminorm on E such that ${}^tTq(x)=q(T(x))$. We shall use the following definitions.

DEFINITION 1. A continuous map T from E into F is presurjective iff for any continuous seminorm p on E there exists a continuous seminorm q on F such that for any $g\in F'$ $|{}^tTg|\leqq p$ implies $|g|\leqq q$.

DEFINITION 2. L and M are T-related iff for any $g\in F'$ ${}^tTg\in L'$ implies $g\in M'$.

DEFINITION 3. A continuous map T from E to F is an epimorphism iff it is surjective and open.

DEFINITION 4. The map T from L into Σ has a homogeneous approximation property iff for any x in L such that $Tx=0$ there exists a sequeance $\{x_i\}$ of the elements in E such that $Tx_i=0$ and $\{x_i\}$ converges to x in L.

3. Theorems. Let G be the linear subspace of L consisting of those x such that $Tx\in M$. We can identify G with the subset of $L\times M$ consisting of the pairs (x, Tx), $x\in G$. We provide G with the topology induced by the product topology. Then G is a Frechet space; it is enough to prove that the set $\{(x, Tx)\,|\,x\in G\}$ is closed in $L\times M$. If $\{x_i\}$ is a sequence converging to x in L and such that Tx_i converges to y in M, since L is con-

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tinuously imbedded in Σ, Tx_i must converge to Tx in Σ, hence $y=Tx$ since Σ is Hausdorff. We note that G is continuously imbedded in L and T from G to M is continuous.

THEOREM 1. *If the map T from E to F is presurjective and G and M are T-related, then $T(L) \supset M$.*

Proof. We shall show that $T(G)=M$ from which $T(L)\supset M$ follows. Since $T(G) \subset M$ by definition and since G and M are Frechct, to show that $T(G)=M$, it is enough to show that the map T from G to M is presurjective (cf. [2]). Let p be a continuous seminorm on G. We may identify p with a continuous seminorm on E. Since T from E to F is presurjective, there exists q, a continuous seminorm on F such that for any $f\in F'$ $|{}^tTf|\geq p$ implies $|f|\leq q$. Let $q'=\sup\{|f| \mid f\in F' \ \& \ |{}^tTf|\leq p\}$. Then q' is a continuous seminorm on F since $q'\leq q$. We note that ${}^tTq'\leq p$, that is, for any $x\in E$ $q'(Tx) \leq p(x)$. Let $g\in F'$ such that $|g|\leq q'$. Then $|{}^tTg|\leq p$. Therefore ${}^tTg\in G'$. Since G and M are T-related, this shows that $g\in M'$. Now since M is a Frechet space, and since for any $g\in F'$ such that $|g|\leq q'$ $g\in M'$, this implies q' is a continuous seminorm on M (cf. p. 48[2]). Above arguments shows that for any continuous seminorm p on G there exists a continuous seminorm q' on M such that for any $g\in M'$ $|{}^tTg|\leq p$ implies $|g|\leq q'$. Therefore T from G to M is presurjective and hence is an epimorphism.

COROLLARY. *If the map T from E to F is presurjective, $T(L)\subset M$, T from L into M is continuous, and L and M are T-related, then $T(L)=M$.*

Proof. We note that the space G introduced before is topologically equivalent to L. Hence our corollary follows immediately from the previous theorem.

THEOREM 2. *Assume that $T(E)$ is dense in F, $T(L)\supset M$ and the map T from L to M has a homogeneous approximation property. Then L and M are T-related.*

Proof. Let $g\in F'$ be such that ${}^tTg\in L'$. On M define a linear functional h such that for any $y\in M$ $h(y)={}^tTg(x)$ where $x\in L$ is such that $Tx=y$. Then $h(y)$ does not depend on x. For if $x'\in L$ is also such that $Tx'=y$, we have $T(x-x')=0$. Since T has an homogeneous approximation property, there exists a sequence $\{x_i\}$ in E converging to $x-x'$ in L and such that $Tx_i=0$ for all i. We then have ${}^tTg(x)-{}^tTg(x')=\lim_i {}^tTg(x_i)=0$ since ${}^tTg(x_i)=g(Tx_i)$. Let us go back to the space G introduced before. Since G and M are Frechet spaces, T is an open mapping from G onto M. Therefore if $y_i\to 0$ in M, there exists, for every i, $x_i\in G$ such that $x_i\to 0$ in G, a fortiori in L, and such that $Tx_i=y_i$. But then $h(y_i)={}^tTg(x_i)\to 0$. This proves the continuity of h. Hence $h\in M'$. Since $h(y)={}^tTg(x)$ for any $y\in M$, take in particular $y\in T(E)\subset F$. Then there is $x\in E$ such that $Tx=y$. This yields for any $y\in T(E)$ $h(y)={}^tTg(x)=g(y)$. Since $T(E)$ is dense in M, we conclude that $h=g\in M'$.

COROLLARY. *Assume that the map T from E into F is presurjective, $T(L)\supset M$, and T has a homogeneous approximation property, then L and M are T-related.*

Proof. Since T is presurjective, $T(E)$ is dense in F. Hence the previous theorem completes the proof.

References

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