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A NOTE ON A DIFFERENTIAL MODULES

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1. ABSTRACT

In this paper, we define a differential module and study its properties. In section 2, as for propositions, we research some properties, directsum, isomorphism of factorization, exact sequence of derived modules. And then as for theorem, I try to present the following statement, if the sequence of homomorphisms of differential modules is exact. Then the sequence of homomorphisms of $Z(X)$ is exact, also the sequence of homomorphisms of $Z'(X)$ is exact.

According to the theorem, as for Lemma, we consider commutative diagram between exact sequence of $Z(X)$ and exact sequence of $Z'(X)$. As an immediate consequence of this theorem, we obtain the following result. If M is an arbitrary module and the sequence of homomorphisms of the modules $Z(X)$ is exact, then the sequence of their tensor products with the trivial endomorphism is semi-exact.

2. A DIFFERENTIAL MODULE

DEFINITION 1. A differential module over R, we mean a module X over R together with a given endomorphism $d: X \to X$ of the module X satisfying the condition $d.d=0$

PROPOSITION 2. The quotient module $H(X)=\text{Ker}(d)/\text{Im}(d)$ is called the derived module of the differential module X. For any given lower sequence C of modules over R. Consider the direct Sum

$$X=\sum_{n\in Z} C_n$$

and the restriction $d: X \to X$ of the cartesian product of all the boundary operators $\partial: C_n \to C_{n-1}$ verify $d.d=0$ and establish

$$H(X)=\sum_{n\in Z} H_n(C).$$

Proof. We take
$$(d.d)\ (\sum_{n\in Z} C_n)=d(d(\sum_{n\in Z} C_n))=d(\sum_{n\in Z}\partial(C_n))$$
$$=\sum_{n\in Z}\partial(\partial(C_n))=0$$

because ∂ is the boundary operation consequently $d: X \to X$ We get $d.d=0$ This implies X is a differintial module according to the definition

$$H(X)=\text{Ker}(d)/\text{Im}(d),$$

We have endomorphism $d.d=0$ with $\sum_{n\in Z} C_n \longrightarrow \sum_{n\in Z}\partial(C_n)$

since $\text{Ker}d=\sum_{n\in Z}\text{Ker}(\partial)$ and $\text{Im}\ d=\sum_{n\in Z}\text{Im}(\partial)$

We have $H(X)=\text{Ker}d/\text{Im}d=\sum\text{Ker}\partial/\text{Im}\partial=\sum_{n\in Z}\text{Ker}\partial/\text{Im}\partial=\sum_{n\in Z}\text{Hn}(C)$

PROPOSITION 3. Consider an arbitrary differential module X over R with differentiation

$$d: X \longrightarrow X$$

Let $\quad Z(X) = \text{Ker}(d) \qquad Z'(X) = \text{Coker}(d)$

$\quad B(X) = \text{Im}(d) \qquad B'(X) = \text{Coim}(d)$

(1) The differentiation $d : X \longrightarrow X$ induces an isomorphism

$\delta : B'(X) \approx B(X)$

and admits the following factorization:

$$X \longrightarrow Z'(X) \longrightarrow B'(X) \xrightarrow{\delta} B(X) \longrightarrow Z(X) \longrightarrow X.$$

(2) This factorization of d yields a homomorphism

$d' : Z'(X) \longrightarrow Z(X)$

(3) Establish the equalities

$\text{Coker}(d') = (X) = \text{Ker}(d')$

(4) $0 \longrightarrow H(X) \longrightarrow Z'(X) \longrightarrow Z(X) \longrightarrow H(X) \longrightarrow 0$

is exact sequence.

Proof. (1) Since $d : X \longrightarrow X$ We have $\text{Imd} \subset \text{Kerd}$ Thus is $B(X) \subset Z(X)$

We obtain $B'(X) = \text{Coim}(d) = X/\text{Kerd}$

By the 1st isomorphism theorm it is clear that there exists

an isomorphism $\quad \delta : B'(X) = X/\text{Kerd} \longrightarrow \text{Imd} = B(X)$

(2) Let $Z'(X) = X/\text{Imd}$ and $B'(X) = X/\text{Kerd}$ denote any two quotient module

Thus $X \xrightarrow{\nu} Z'(X) \xrightarrow{\rho} B'(X) \xrightarrow{\delta} B(X) \xrightarrow{i} Z(X) \xrightarrow{j} X$

factorize where ν is the canonical map and i, j are inclusion maps and ρ is a map sending $X+\text{Imd}$

to $X+\text{Kerd}$ Thus we get a map $d' = i.\delta.\rho : Z'(X) \longrightarrow Z(X)$ This map is homomorphism.

(3) $\text{Cokerd}' = Z(X)/\text{Imd}' = Z(X)/\text{Im}(i\cdot\delta\cdot\rho) = Z(X)/\text{Im}(j\cdot i\cdot\delta\cdot\rho\cdot\nu)$

because ν, j are epimorphism and monomorphism respectively.

The latter term $Z(X)/\text{Im}(j\cdot i\cdot\delta\cdot\rho\cdot\nu) = Z(X)/\text{Imd} = H(X)$

Similarly $H(X) = \text{Ker}(d')$

(4) $0 \longrightarrow H(X) \xrightarrow{\alpha} Z'(X) \xrightarrow{d'} Z(X) \xrightarrow{\beta} H(X) \longrightarrow 0$

$\alpha : H(X) = \text{Coker}(d) = Z(X)/\text{Imd} \quad X/\text{Ind} = \text{Coker}(d') = Z'(X)$ is an imbedding.

$\beta : \text{Kerd} = Z(X) \quad Z(X)/\text{Imd} = H(X)$ is canonical map

To show the above sequence is exact for any element $x + \text{Imd} \in Z(X)/\text{Imd}$.

We have $(d'.\alpha)(x+\text{Imd}) = d'(x+\text{Imd}) = \bar{O}$

Thus $\text{Im}\alpha \subseteq \text{Kerd}'$

Conversely Let $x + \text{Imd} \in Z'(X)$

then there is an element $a \in H(X)$ with $\alpha(a) = x + \text{Imd} \qquad d'(x+\text{Imd}) = \bar{O}$

by (2) we get $\text{Im}\alpha = \text{Kerd}'$.

Similary for any element $x + \text{Imd} \in Z'(X)$.

We have $(\beta.d')(x+\text{Imd}) = \beta[Z(x)] = O$

because $\beta : \text{Ker}(d) = Z(X) \longrightarrow Z(X)/\text{Imd} = H(X)$ is canonical map.

Conversely proof well do to.

Consequensely $\text{Imd}' = \text{Ker}\ \beta$, we proved the exact sequence.

THEOREM 4. If the sequence $\quad O \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C$

of homomorphisms of differential modules (which commute with the differentiations d) over R

is exact.

Let $Z(A)=\mathrm{Ker}d.$ $Z(B)=\mathrm{Ker}d'$ $\mathrm{Ker}(C)=\mathrm{Ker}d''$,

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ d\downarrow & & d'\downarrow & & d''\downarrow \\ A & \longrightarrow & B & \longrightarrow & C \end{array}$$

then $O\longrightarrow Z(A)\xrightarrow{\alpha^*}Z(B)\xrightarrow{\beta^*}Z(C)$ is exact.

Proof. For any element $a\in Z(A)$ we have $d(a)=0$.

Since $(\beta^*\cdot\alpha^*)$ $(a)=\beta^*(\alpha^*(a))=\beta^*(\alpha(a))=\beta(\alpha(a))=(\beta\cdot\alpha)(a)=0$ is the trivial homomorphism, we have $\mathrm{Im}\alpha^*\subset\mathrm{Ker}\ \beta^*$.

On the other hand, for arbitrary element $b\in Z(B)$ and $b\in$ Ker β^* we have $\beta^*(b)=\beta(\mathrm{b})=0$.

Since $\beta(b)=0$, this implies $b\in\mathrm{Ker}\ \beta=\mathrm{Im}\alpha$.

There exists an element $a\in A$ with $\alpha(a)=b$. By the commutativity of the left square, we have $\alpha(d(a))=(\alpha\cdot d)$ $(a)=(d'\cdot\alpha)$ $(a)=d'(\alpha(a))=d'(b)=0$.

Since α is a monomorphism, this implies $d(a)=0$.

Hence we obtain $a\in Z(A)$, we get Ker $\beta^*\subset\mathrm{Im}\alpha^*$.

We have $\mathrm{Im}\alpha^*=\mathrm{Ker}\ \beta^*$. This completes the proof.

THEOREM 5. If the sequence $A\xrightarrow{\alpha}B\xrightarrow{\beta}C\longrightarrow O$

of homomorphisms of differential modules (which commute with the differentiations d) over R is exact.

Let $Z'(A)=A/\mathrm{Imd}$ $Z'(B)=B/\mathrm{Imd}'$ $Z'(C)=C/\mathrm{Imd}''$,

Then $Z'(A)\xrightarrow{\mu}Z'(B)\xrightarrow{\nu}Z'(C)\longrightarrow O$ is exact.

Proof. Let us consider arbitrarily homomorphism μ and ν of quotient module over

$$\mathrm{Coker}(d)\xrightarrow{\mu}\mathrm{Coker}(d') \text{ and } \mathrm{Coker}(d')\xrightarrow{\nu}\mathrm{Coker}(d'').$$

Then we must make sure that μ,ν is well defined.

If $a_1+\mathrm{Imd}=a_2+\mathrm{Imd}$ then $a_1-a_2\in\mathrm{Imd}$

there exists element $a\in A$ with $d(a)=a_1-a_2$.

By the commutativity of left square, we have $d'\alpha(a)=\alpha\ d(a)$

Then we have

$\mathrm{Im}\ d'\ni d'(\alpha(a))=\alpha(a_1-a_2)=\alpha(a_1)-\alpha(a_2)$

This implies $\alpha(a_1)+\mathrm{Imd}'=\alpha(a_2)+\mathrm{Imd}'$.

Hence we obtain $\mu(a_1+\mathrm{Imd})=\mu(a_2+\mathrm{Imd})$ consequently well defined μ. Similary we can define ν homomorphism.

Thus we have established μ-homomorphism such that

$$\mu:\mathrm{Coker}(d)\longrightarrow\mathrm{Coker}(d')$$

satisfying $a+\mathrm{Imd}\longrightarrow\alpha(a)+\mathrm{Imd}'$.

Here we must be established

$$\mu[(a_1+\mathrm{Imd})+(a_2+\mathrm{Imd})]=\mu[(a_1+a_2)+\mathrm{Imd}]$$
$$=\alpha(a_1+a_2)+\mathrm{Imd}'=\alpha(a_1)+\alpha(a_2)+\mathrm{Imd}'=[\alpha(a_1)+\mathrm{Ind}']+[\alpha(a_2)+\mathrm{Imd}']$$

$=\mu(a_1+\mathrm{Imd})+\mu(a_2+\mathrm{Imd})$.

Also $\mu[r(a+\mathrm{Imd})]=\mu(ra+\mathrm{Imd})=\alpha(ra)+\mathrm{Imd}'=r[\alpha(a)+\mathrm{Imd}']=r\mu(a+\mathrm{Imd})$.

Hence we get that μ is module-homomorphism.

Similarly we can define that ν is module homomorphism consequently we well define module homomorphisms.

$$\mathrm{Coker}(d)\xrightarrow{\mu}\mathrm{Coker}(d')\xrightarrow{\nu}\mathrm{Coker}(d'').$$

The last assertion in theorem is to show exact sequence.

Let $a+\mathrm{Imd}\in\mathrm{Cokerd}$, be arbitrarily given.

We have $\nu\cdot\mu\ (a+\mathrm{Imd})=\nu[\mu(a+\mathrm{Imd})]=\nu[\alpha(a)+\mathrm{Imd}']=\beta\alpha(a)+\mathrm{Imd}''=O+\mathrm{Imd}''=\bar{O}$.

We get $\mathrm{Im}\mu\subset\mathrm{Ker}\ \nu$.

Conversely, let $b+\mathrm{Imd}'\in\mathrm{Ker}\ \nu$ then we can easily verify that

$\nu(b+\mathrm{Imd}')=\bar{O}$ $\beta(b)+\mathrm{Imd}''=\bar{O}$. We have $\beta(b)\in\mathrm{Imd}''$.

Hence there is an element $c\in C$ with $\beta(b)=d''(c)$.

Since β is a epimorphism, there exists an element $c\in C$ and $b\in B$ with $\beta(b)=c$.

By the commutativity of the right square

we have $\beta[b-d'(b)]=\beta(b)-\beta d'(b)=\beta(b)-d''\beta(b)=\beta(b)-d''(c)=\beta(b)-\beta(b)=0$.

This implies $b-d'(b)\in\mathrm{Ker}\beta=\mathrm{Im}\alpha$.

Hence there exists an element $a\in A$ with $\alpha(a)=b-d'(b)$.

In this case $\alpha(a)-b\in\mathrm{Imd}'$.

We have $\alpha(a)+\mathrm{Imd}'=b+\mathrm{Imd}'$. This implies $\mu(a+\mathrm{Imd})=b+\mathrm{Imd}'$, since $b+\mathrm{Imd}'\in\mathrm{Im}\ \mu$.

We get $\mathrm{Ker}\ \nu\subset\mathrm{Im}\ \mu$.

This completes the proof $\mathrm{Im}\ \mu=\mathrm{Ker}\ \nu$. We obtain exact.

LEMMA 6. We obtain a commutative diagram

$$\begin{array}{ccccccc} & & Z'(A) & \longrightarrow & Z'(B) & \longrightarrow & Z'(C) & \longrightarrow 0\\ & & \downarrow d' & & \downarrow d' & & \downarrow d' & \\ 0 & \longrightarrow & Z(A) & \longrightarrow & Z(B) & \longrightarrow & Z(C) & \end{array}$$

of homomorphisms of modules with exact rows.

Proof. Let us define arbitrarily homomorphisms d and d' with $d'(a+\mathrm{Imd})=d(a)$

oevr $A/\mathrm{Imd}\xrightarrow{d'}\mathrm{Kerd}$

If $a_0+\mathrm{Imd}=a_1+\mathrm{Imd}$ then $a_0-a_1\in\mathrm{Imd}\subseteq\mathrm{Kerd}$.

Since $d^2=0$, we have $d(a_0-a_1)=0$ $d(a_0)-d(a_1)=0$ $d(a_0)=d(a_1)$

Hence, we can define d'.

Here we must be established module-homomorphism d'

$d'[(a_0+\mathrm{Imd})+(a_1+\mathrm{Imd})]=d'(a_0+a_1+\mathrm{Imd})=d(a_0)+d(a_1)$

$=d'(a_0+\mathrm{Imd})+d'(a_1+\mathrm{Imd})$

also, $d'[r(a_0+\mathrm{Imd})]=d'(ra_0+\mathrm{Imd})=d(ra_0)=rd'(a_0+\mathrm{Imd})$.

Hence we get that d' is module-homomorphism.

To show the diagram

$$\begin{array}{ccc} Z'(A) & \xrightarrow{\alpha_*} & Z'(B)\\ \downarrow d' & \bar{\alpha} & d' \downarrow\\ Z(A) & \longrightarrow & Z(B) \end{array}$$ is commutative.

For any element $a+\text{Imd}\in Z'(A)$

there exists $(d'.\alpha_*)(a+\text{Imd})=d'(\alpha(a)+\text{Imd})=d'(\alpha(a)+\text{Imd})=d(\alpha(a))$.

On the other hand $\alpha.d'(a+\text{Imd})=\alpha(d(a))=d(\alpha(a))$.

Similarly we can prove that

$$\begin{array}{ccc} Z'(B) & \xrightarrow{\beta_*} & Z'(C) \\ \downarrow d' & & \downarrow d' \\ Z(B) & \xrightarrow{\bar{\beta}} & Z(C) \end{array}$$ Is commutative.

Consequently we get commutative diagram.

THEOREM 7. *An* exact sequence

$$O\longrightarrow Z(A)\xrightarrow{f}Z(B)\xrightarrow{g}Z(C)\longrightarrow O$$

Then, $O\longrightarrow Z(A)\otimes M\xrightarrow{f_*}Z(B)\otimes M\xrightarrow{g_*}Z(C)\otimes M\longrightarrow O$

is semi-exact if M is an arbitrary moule over R and j is trivial endomorphism.

Proof. We prove, for any two consecutive homomorphisms f_* and g_*, g_*f_* is trivial homomorphism.

We define f_* and g_* as follows

$$f_*=f\otimes j,\qquad g_*=g\otimes j$$

We claim $g_*\ f_*$ is trivial homomorphism.

For arbitrary $t\in Z(A)\otimes M$, let $t=\sum_{k=1}^{n}(a_k\otimes m_k)$ for $a_k\in Z(A)$ and $m_k\in M$ for each $k=1,2,\cdots\cdots n$,

Then we have by tensor product

$$g_*.f_*(t)=g_*(f\otimes j)\left(\sum_{k=1}^{n}(a_k\otimes m_k)\right)$$

$$=g_*\left(\sum_{k=1}^{n}(f(a_k)\otimes j(m_k)\right)$$

And we apply the same method and tensor product

$$g_*.f_*(t)=\sum_{k=1}^{n}(g.f)(a_k)\otimes(j.j)(m_k)$$

$$=e_{Z(A)}\otimes e_M.$$

It means the unit element of $Z(A)\otimes M$

This completes the proof.

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