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A note on H-closed spaces

by

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1. INTRODUCTION.

In this paper I want to find out the properties which is satisfied in H-closed space. Many characterizations was introduced by Larry L. Herrington and Paul E. Long with weakly continuous mapping [4] and strongly closed graph, filterbase, family of regular-closed sets and nets [5].

Here I calculate the properties that a continuous surjection image of H-closed space is H-closed, a preimage of continuous bijection of H-closed and Urysohn space is H-closed space, a H-closed subspace of Hausdroff space is closed and if product space is H-closed space then each projection space is H-closed but not for converse.

Moreover all the notation are based on [1].

2. PRELIMINARY AND DEFINITION.

Definition 2.1. A Hausdroff space X is H-closed if for every $\{U_\alpha|\alpha\in\Delta\}$ there exists a finite subfamily $\{U_{\alpha i}|\ i=1,2,\cdots\cdots,n\}$ such that $\bigcup_{i=1}^{n}$ Cl $(U_{\alpha i})=X$.

Definition 2.2. A mapping $f: X\to Y$ is said to be weakly continuous (briefly w.c.) if for each point $x\in X$ and each open set $V\subset Y$ containing $f(x)$, there exists an open set $U\subset X$ containing x such that $f(U)\subset \mathrm{Cl}(V)$.

We omitt the other definition.

Lemma 2.3. (Levine). A mapping $f: X\to Y$ is w.c. iff for each open set $V\subset Y$, $f^{-1}(V)\subset \mathrm{Int}(f^{-1}(\mathrm{Cl}(V)))$.

Proof. [5].

Lemma 2.4 If Y is a Urysohn space and $f: X\to Y$ is w.c. injection, then X is Hausdorff.

Proof. For any distinct points $x_1, x_2\in X$, we have $f(x_1)\neq f(x_2)$ becouse f is injective. Since Y is Urysohn, there exist open sets V_1 and V_2 in Y such that $f(x_1))\in V_1$, $f(x_2)\in V_2$ and $\mathrm{Cl}(V_1)\cap \mathrm{Cl}(V_2)=\phi$. Hence we have $\mathrm{Int}(f^{-1}(\mathrm{Cl}(V_1)))\cap \mathrm{Int}(f^{-1}(\mathrm{Cl}(V_2)))=\phi$. Since f is w.c., by L. 2.3., we have $x_j\in f^{-1}(V_j)\subset \mathrm{Int}(f^{-1}(\mathrm{Cl}(V_j)))$ for $j=1,2$. This implies that X is Hausdroff.

3. MAIN PROPERTIES.

Theorem 3.1. A continuous surjection image of H-closed space is H-closed.

Proof. Let $f: X\to Y$ be a continuous surjection and X be a H-closed space, $\{U_\alpha|\alpha\in\Delta\}$ be arbitrary open cover of Y then there exists open cover of X with $\{f^{-1}(\bigcup_\alpha)|\alpha\in\Delta\}$ since f is continuous. On the other hand since X is H-closed space there exists finite subcover $\{f^{-1}(U_{\alpha i})|i=1,2,\cdots\cdots,n\}$ of $\{f^{-1}(U_\alpha)|\alpha\in\Delta\}$ such that $\bigcup_{i=1}^{n}$ Cl $(f^{-1}(U_{\alpha i}))=X$. Thus, for above finite subcover, $\{U_{\alpha 1}, U_{\alpha 2},\cdots\cdots, U_{\alpha n}\}$ is finite subcover of $\{U_\alpha|\alpha\in\Delta\}$ and moreover $Y=f(X)=f(\bigcup_{i=1}^{n}$ Cl $(f^{-1}(U_{\alpha i})))\subset f(\bigcup_{i=1}^{n} f^{-1}(\mathrm{Cl}(U_{\alpha i})))=$

$(f^{-1}(\overset{n}{\underset{i=1}{U}}\mathrm{Cl}\ (U_{\alpha i})))=\overset{n}{\underset{i=1}{\cup}}\ (\mathrm{Cl}(U_{\alpha i}))$. Clearly $\overset{n}{\underset{i=1}{U}}\ (\mathrm{Cl}(U_{\alpha i}))\subset Y$. Thus $\overset{n}{\underset{i=1}{U}}\mathrm{Cl}(U_{\alpha i})=Y$. This implies Y is H-closed space.

Theorem 3.2. A preimage of continuous bijection of H-closed and Urysohn space is H-closed space.

Proof. We put $f: X\to Y$ be continuous bijection and Y is H-closed and Urysohn space, then by Lemma 2.4 X is Hausdorff since generally if f is continuous then f is w.c. [4].

If each open cover $\{U_\alpha|\alpha\in\Delta\}$ of X is the preimage of some open cover of Y of the form $\{V_\alpha|\alpha\in\Delta\}$ Thus $\{f^{-1}(V_\alpha)|\alpha\in\Delta\}$ be a cover of X such that $f^{-1}(V_\alpha)=U_\alpha$ for each $\alpha\in\Delta$. Since Y is H-closed there exist finite subcover $\{V_{\alpha i}|i=1,2,\cdots\cdots,n\}$ of $\{V_\alpha|\alpha\in\Delta\}$ such that $\overset{n}{\underset{i=1}{\cup}}\mathrm{Cl}\ (V_{\alpha j})=Y$. For this finite subcover $\{f^{-1}(V_{\alpha i})|i=1,2,\cdots\cdots,n\}$ is the finite subcover of $\{f^{-1}(V_\alpha)|\alpha\in\Delta\}$ and that $\overset{n}{\underset{i=1}{\cup}}\mathrm{Cl}\ (f^{-1}(V_{\alpha i}))\subset\overset{n}{\underset{i=1}{\cup}}f^{-1}(\mathrm{Cl}(V_{\alpha i}))=f^{-1}(\overset{n}{\underset{i=1}{\cup}}(\mathrm{Cl}(V_{\alpha i})))=f^{-1}(Y)=X$. Since f is surjection, Suppose if $x\in X$, $x\notin\overset{n}{\underset{i=1}{\cup}}(\mathrm{Cl}(f^{-1}(V_{\alpha i})))$, then $x\notin\overset{n}{\underset{i=1}{\cup}}f^{-1}(\mathrm{Cl}(V_{\alpha i}))=f^{-1}(\overset{n}{\underset{i=1}{\cup}}\mathrm{Cl}(V_{\alpha i})=f^{-1}(Y)=X$ since f is surjection. $f(x)\notin Y$ Thus this implies X is H-closed.

Theorem 3.3. A H-closed subspace of Hausdroff space is closed.

Proof. Let X be Hausdroff space and $A\subset X$ be the H-closed subspace, then as for the relative topology, we pick any open covering $\{U(a)\cap A|a\in A\}$ of A. On the other hand, since $U(a)$ is open in Hausdroff space and A is H-closed space itself, there exists finite subcover $\{U(a_i)\cap A|i=1,2,\cdots\cdots,n\}$ of $\{U(a)\cap A|a\in A\}$ such that $\overset{n}{\underset{i=1}{\cup}}\mathrm{Cl}\ (U(a_i)\cap A)=A$. Since finite union of closed sets is closed, thus A is closed.

Theorem 3.4. Let $\{Y_\alpha|\alpha\in\Delta\}$ be a family of spaces. If $\prod_\alpha Y_\alpha$ is H-closed space, then Y_α is H-closed for each $\alpha\in\Delta$.

Proof. We take projection $p_\beta: \prod_\alpha Y_\alpha\to Y_\beta$, since projection is continuous surjection and by theorem 3.1 Y_β is H-closed space.

In above theorem 3.4, converse is not hold by property of theorm 3.2.

References

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요 약

남정완, 배철곤, 민강주

L.L. Herrington 과 P.E. Long 이 서술한 H-closed Space 에 대해서 성질 즉 H-closed Space 의 연속이고 전사인 상은 H-closed Space 가 된다는 사실과 그외 몇가지 성질을 조사 했다.