韓國數學教育學會誌<數學教育>
1975. 12 Vol. XIV No. 1

Journal of the Korea Society of Mathematical Education
Dec. 1975 Vol. XIV No. 1

On Nearly Compact Spaces

by

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In this paper, I introduce some notions weaker than compactness, investigate some properties of this notions, and find the relations between them.

Given atopological property P a P-space X is called P-closed provided X is a closed subset in every P-space in which it can be imbedded. let us denote H-closed for Hausdorff-closed, for regular T_1-closed spaces.

Lemma 1. If X is a Hausdorff space, the following conditions are equivalent;

(a) X is H-closed.

(b) Every open filterbase on X has at least one adherent point.

(c) Every open covering of X contains a finite dense subcovering.

Proof. (a) implies (b). If there is open filterbase $\mathfrak{F}$ with no adherent point, choose a now point P not in X and take $Y=X\cup\{p\}$ with the topology generated by $T\cup\{U\cup\{p\} : U\in\mathfrak{F}\}$ where T is the topology on X. Then Y is a Hausdorff space and X is not closed in Y.

(b) implies (c). Suppose there is an open covering $\mathfrak{U}$ of X such that $\mathfrak{U}$ contains no finite dense subcovering. Then $\mathfrak{F}=\{(C_1U)^c : U\in\mathfrak{U}\}$ is an open filterbase which has no adherent point.

(c) implies (a) Suppose Y is a Hausdorff space contains X and X is not closed in Y. Take a limit point p of X in Y which is not an element of X, and for each x in X disjoint neighborhoods V_x, U_x of x and p respectively. Then $\{V_x\cap X : x\in X\}$ is an open covering of X which has no finite dense subcovering.

By condition (c) we obtain the following;

Corrollary. A regular T_1-space is H-closed if it is compact.

Example. Let X be the unit interval of R with the topology generated by the open intervals and Q the rationals. Then

(i) X is H-closed and not regular.

(ii) X is completely Hausdorff space, hence it has the Stone-Weierstrass property by 6.7 of [6].

(iii) X-$Q\subset X$ is closed subset of H-closed space which is not H-closed.

H-closed subspace of Hausdorff space is closed, but the converse is false as the above example. In fact it is an unsolved problem to find a necessary and sufficient condition for a subset of an H-closed space to be H-closed. Only I know is that regular closed subset (i,e. the closed subset that is equal to the closure of its interior) of H-closed space is H-closed.

An open filterbase of a topological space is called regular filterbase if it is equivalent with a closed filterbase; that is an open filterbase in which each set contains the closure of some member of the filterbase.

Lemma 2. For regular T_1-space X, the followings are equivalent;

(a) X is R-closed.

(b) Every regular filterbase on X has at least one adherent point.

(c) Every maximal regular filterbase is convergent.

Proof. (a) implies (b). It is similar to the proof of lemma 1.

(b) iff (c). It can be proved by a routine manner.

(b) implies (a). Suppose X is not R-closed, then there is a regular T_1-space Y such that $X \subset Y$ and X is not closed in Y. Take a point $p \in CIX - X$ and let ν be the complete neighborhood system of p. Then by the regularity of Y we know that ν is a regular filterbase in Y and $X \cap \nu$ is also a regular filterbase in X, and it has no adherent point.

Lemma 3. If X is R-closed, then each countable open filterbase in X has an adherent point.

Proof. let $\mathfrak{U} = \{U_1, U_2, \cdots\cdots\}$ be any countable open filterbase in X, and without loss of generality assume that $U_n \supset U_{n+1}$. There exist regular filterbases that are coaser than $\mathfrak{U}$ (e.g. $\{X\}$), hence by Zorn's lemma there is a maximal regular filterbase $\mathfrak{M}$. From lemma 2, we have an adherent point p of $\mathfrak{M}$, and p must be also an adherent point of $\mathfrak{U}$.

Because if p is not an adherent point of $\mathfrak{U}$ there is an m such that for all $n \geq m$, p is not in the closure of U_n. By regularity of X, there is an open neighborhood G of p such that whose closure does not meet with CIU^n, n m. Since $(CIG)^C$ is regular open, the collection $(CIG)^C \cap \mathfrak{M}$ is a regular filterbase which is strictly finer than $\mathfrak{M}$ and coaser than $\mathfrak{U}$. This contradicts the maximality of $\mathfrak{M}$. This completes the proof.

A space X is called feebly compact (or lightly compact) if each nbd-finite family of open subsets of X is finite.

Lemma 4. A space X is feebly compact if and only if every countable open filterbase in X has an adherent point.

Proof. (necessity) If for some countable open filterbase $\mathfrak{F} = \{G_n\}$, ad $\mathfrak{F} = 0$. We may suppose without loss of generality $G_n \supset G_n + 1$ for every n. Since ad $\mathfrak{F} = 0$, is a nbd finite family. Hence by hypothesis $\mathfrak{F}$ must be finite and it is a filterbase. Hence ad $\mathfrak{F} = \cap CIG_n \neq 0$. This contradiction shows that every countable open filterbase has nonvoid adherence.

(sufficiency) Suppose $\mathfrak{U}$ is a nbd-finite family of open subsets of X. If $\mathfrak{U}$ is not finite we can choose a countable infinite subset $\{U_n\} \subset \mathfrak{U}$. Consider the collection $\mathfrak{F}$ of all sets $V_n = \bigcup_{i=n}^{\infty} U_i$, $n = 1, 2, \cdots\cdots$, then $\mathfrak{F}$ is a countable open filterbase in X. And by hypothesis $\mathfrak{F}$ has an adherent point p. Since each neighborhood of p meets with every member of $\mathfrak{F}$, each neighborhood of p meets with infinitely many $U_n's$. Hence $\mathfrak{U}$ is not nbdfinite at p.

From lemma 1 and 2 we have the results;

Theorem 1. (a) Every R-closed space is feebly compact.

(b) Every H-closed space is feebly compact.

Theorem 2. If X is countably compact, it is feebly compact.

Proof. Suppose X is countably compact and there is a countable open filterbase $\mathfrak{F} = \{U_n\}$ that has no adherent point. Then the collection $\mathfrak{U}$ of all complements of closures of U_n is a countable open cover of X and since X is countably compact $\mathfrak{U}$ has a finite subcover $\{(CIU_1)^C, (CIU_2)^C, \cdots\cdots, (CIU_n)^C\}$ and $\bigcup_{i=1}^{n} (CIU_i)^C = (\bigcap_{i=1}^{n} CIU_i)^C = X$. Hence $\bigcap_{i=1}^{n} CIU_i = 0$. But it contradicts to the fact that $\mathfrak{F}$ is a filterbace.

Therefore $\mathfrak{F}$ must have nonvoid adherence.

Every sequentialy compact space is countably compact and hence feebly compact. A space is called χ_0-bounded if every countable subset has compact closure.

Theorem 3. If X is χ_0 -bounded, it is sequentialy compact.

Proof. If $\{x_n\}$ is a sequence in χ_0 -bounded space X, then $C=\mathrm{Cl}\{x_n\}$ is compact, and the sequence $\{x_n\}$ in compact space C has a convergent subsequence.

By this theorem and the previous remark we may conclude that;

Theorem 4. Every χ_0 -bounded space is feebly compact.

A Hausdorff space is called Lindelöf space if each open covering contains a countable subcovering.

Theorem 5. If X is Lindelöf and feebly compact, X is H-closed.

Proof. Using lemma 1, we prove the theorem by showing that each open covering has a finite dense subcovering. Let Q be any open covering of X, then Q has a countable subcovering $\mathfrak{U}=\{U_1, U_1, \cdots\}$ since X is a Lindelöf space. If $\mathfrak{U}$ has no finite dense subcovering, the collection v of all sets $V_n=(\bigcup_{i=1}^{n} CIU_i)^C$, $n=1, 2, \cdots$, is a countable open filterbase in X. Since X is feebly compact, v has an adherent point by lemma 4. On the other hand, for each point x in X there is at least on U_n containing x, and U_n is an open set not meeting V_n. Hence x is not an adherent point of v. It is contradiction. Therefore $\mathfrak{U}$ and Q must have a finite dense subcovering. It completes the proof.

Theorem 6. If X is Lindelöf and χ_0-bounded, it is compact.

Proof. By theorem 3 and the remark above theorem 3 the x_0-bounded space is countable compact. And the countably compact Lindelöf space is compact. This proves theorem.

Let Ω be the first uncountable ordinal number then by [1],

Lemma 5. Every countable subset of

$$[0, \Omega[=\{x; x \text{ is an ordinal, } 0\leq x<\Omega\}$$

has an upper bound in $[0, \Omega[$.

Example. Let Y be the space $[0, \Omega[$ with the order topology then as a subspace of the compact space $[0, \Omega]$, Y has the properties;

(i) Y is feebly compact by lemma 5 and theorem 2.

(ii) Y is normal, hence regular.

(iii) Y is neither H-closed nor R-closed.

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