構造用 샌드위치板의 휨特性에 對하여(Ⅱ)

Flexural Behavior of Structural Sandwich Panels(Ⅱ)

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Table-1 conitnud

Sample name	P_1/w_1 lb/in	P_2/w_2 lb/in	P_{1yld} lb	P_{1max} lb	Mode of failure
Urecomb Core					
PUCAL—1	2,040	3,540	396	510	C
PUCAL—2	2,230	3,720	528	576	C
PUCFG—1	860	1,320	360	522	C
PUCFG—2	1,200	1,560	288	348	C
PUCGS—1	1,990	3,060	240	450	D
PUCGS—2	2,280	2,760	606	606	D
MUCAL—1	1,920	2,820	432	516	C
MUCAL—2	1,300	2,340	384	540	C
PUCP—1	1,000	2,760	444	570	E
PUCP—2	1,560	4,660	636	912	A
PUCP—3	1,510	3,360	576	—	B
PUCP—4	1,260	2,160	324	576	B
PUCP—5	960	1,500	168	306	B
PUCP—6	1,380	2,220	192	420	B
PUCP—7	2,760	3,720	360	408	A
PUCP—8	2,160	4,320	276	456	A
FGUCFG—1	300	420	372	394	C
FGUCFG—2	283	420	360	390	C
FGUCFG(R)—1	607	926	636	660	D
FGUCFG(R)—2	630	960	618	618	D
PUCP(R(—1	1,650	3,624	576	696	E
PUCP(R)—2	1,800	3,600	1,272	1,332	F

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Table-2. Flexural Stiffnesses, D, Calculated by Equation(2)

Sample name	D lb—in²
PSUAL	5,952,168
PSUFG	1,678,952
PSUGS	8,605,410
MSUAL—1	2,287,156
MSUAL—2	4,112,000
PSUP	7,076,250
FGSUFG—1	2,133,378
FGSUFG—3	4,694,530
PPCP	36,767,812
PUCAL	9,751,470
PUCFG	2,833,000
MUCAL—1	4,330,097
MUCAL—2	6,992,924
PUCP	11,666,250
PUCP—7	24,288,750
FGUCFG	2,133,378
FGUCFG(R)	,1736,373
PUCP(R)	7,076,250

Table-3. Modulus of Rigidity

Sample name	Modulus of rigidity, G_c, psi	
	Midpoint test	Quarter point test
PSUAL—1	365	203
PSUAL—2	203	172

PSUFG—1	170	159
PSUFG—2	209	167
PSUGS—1	173	99
PSUGS—2	305	229
MSUAL—1	397	—
MSUAL—2	179	260
PSUP—1	320	223
PSUP—2	340	237
PSUP—3	187	194
PSUP—4	162	141
PSUP—5	89	103
PSUP—6	137	137
FGSUFG—1	123	97
FGSUFG—2	99	88
FGSUFG—3	100	108
FGSUFG—4	128	120
PPCP—1	113	125
PPCP—2	113	125
PUCAL—1	964	904
PUCAL—2	1,093	974
PUCFG—1	483	385
PUCFG—2	921	526
PUCGS—1	835	650
PUCGS—2	989	573

Table 3. Modulas of Rigidity(continued)

Sample name	Modulus of rigidity, G_c, psi	
	Midpoint test	Quarter-point test
MUCAL—1	1,729	1,288
MUCAL—2	591	577
PUCP—1	333	520
PUCP—2	560	1,085
PUCP—3	539	674
PUCP—4	434	384
PUCP—5	318	251
PUCP—6	483	397
PUCP—7	714	475
PUCP—8	538	566
FGUCFG—1	139	97
PGUCFG—2	130	97
FGUCFG(R)—1	436	348
FGUCFG(R)—2	468	373
PUCP(R)—1	849	1,003
PUCP(R)—2	966	1,196

The highest value for the midpoint load test in MUCAL should obviously be discarded, considering the agreement of the values for the suter quarter-point loadtests within the nther opecimen combinations.

The general outlook in Figs. 6 and 7 shows that the moduli of rigidity obtained have considerable variations for each type of core material. Possible sources of variation are considered to be:

(1) nonhomogeneity of component materials,

(2) relative slip between the facing and core at the glue line,

(3) the possibility of movement of the neutral axis because of the low rigidity of the core,

(4) compression of the core in the direction of load, and

(5) experimental error.

Combining the results ef tests the mean core shear moduli are 180 psi for the panels having solid polyurethane foam core and 630 psi for the panels having urecomb core.

From the values of the moduli of rigidity for the specimens in which the facing material of fiberglas is involved, two facts can be stated. First, the modulus of elasticity, E, in the computation of D for the specimen with a facing of fiberglas seems to be some that too high. Second, the materials and fabrication of these specimans were the most uniform of all the samples tested. It is shown in the comparison between the specimens fabricated with the normal glue and those with a special rigid glue that the rigidity of the glue line improves the bending stiffness of the sandwich panel system as a whole by improving the effective value of G_c.

Alternatively, values of G_c were calculated by Equation (8) which is a simplified formula. These results are presented in Table 7. The values of G_c in Table 7 are higher than those in Table 6 by from 13 to 34 per cent. The ratios of the G_c value based on midpoint load tests and those based on quarter-point load tests were found to be, mostly, close to the number 1.0.

Another method used to find the bending stiffness, D, and the shear modulus, G_c, was the solution of the two simultaneous equations·

$$w_1=\frac{P_1a^4}{48D}+\frac{P_1a}{4N} \quad \text{for midpoint loading}$$

$$w_2=\frac{11P_2a^4}{768D}+\frac{P_1a}{8N} \text{for quarter-point loading.}$$

These two equations are derived from Eq. (8) by using appropriate numbers from Table 1 for K_b and K_s, depending upon the loading conditions, midpoint load test or outer quarter point load test. This sttempt, however, is considered to be inappropriate for these experimental results. If Eq. (8) is exactly representative of the behauior of the construction, theoretically a unique solution of the two equations can be obtained within the range of the ratio of slopes P_1/w_1 and P_2/w_2 which lies between 0.5 and

Table-4. Modulas of Rigidity
(by simplified formula)

Sample name	Modulus of rigidity, G_c, psi	
	Midpoint test	Quarter-point test
PSUAL—1	440	243
PSUAL—2	243	207
PSUFG—1	211	198
PSUFG—2	260	209
PSUGS—1	206	118
PSUGS—2	364	274
MSUAL—1	464	—
MSUAL—2	217	320
PSUP—1	429	299
PSUP—2	322	318
PSUP—3	251	261
PSUP—4	217	189
PSUP—5	120	138
PSUP—6	184	183
FGSUFG—1	134	106
FGSUFG—2	107	97
FGSUFG—3	106	115
FGSUFG—4	135	128
PPCP—1	128	142
PPCP—2	128	142
PUCAL—1	1,125	1,061
PUCAL—2	1,279	1,146
PUCFG—1	581	465
PUCFG—2	1,147	645

Table-4. continued

Sample name	Modulus of rigidity, G_c, psi	
	Midpoint test	Quarter-point test
PUCGS—1	962	750
PUCGS—2	1,142	660
MUCAL—1	2,237	1,674
MUCAL—2	700	692
PUCP—1	418	652
PUCP—2	703	1,361
PUCP—3	676	845
PUCP—4	545	482
PUCP—5	399	315
PUCP—6	607	498
PUCP—7	834	555
PUCP—8	628	662
FGUCFG—1	153	106
FGUCFG—2	142	106
FGUCFG(R)—1	510	410
FGUCFG(R)—2	552	445
PUCP(R)—1	1,140	1,346
PUCP(R)—2	1,927	1,606

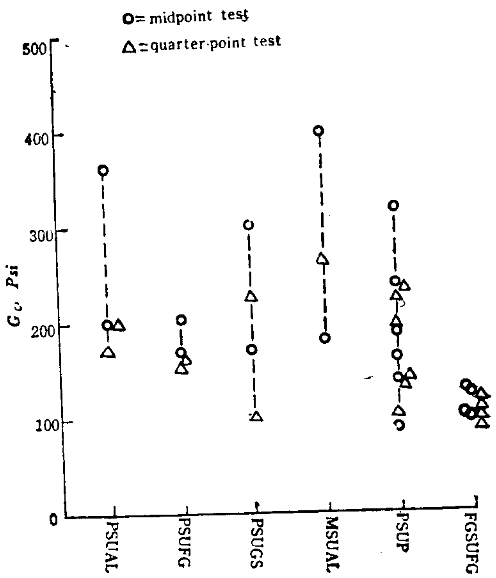

Fig. 6. Modulus of Rigidity of Core for Sampls with the Core of solid Urethane

0.688. However, as the ratio approaches the value 0.5, D approaches infinity and as the ratio approaches 0.688, G_c approaches infinity. Neither D nor G_c, in reality, can approach infinity. In many cases of these specimens the test result indicates these ratios are near or outside of the extreme limits. The principal reason for this problem is that the equations are such that the effects of experimental error are magnified by the computations.

In an attempt to check the range given in the analysis of Hoff and Mautner (2), the values pa/2 were computed as given in Table 8. In the computation of p defined byEq. (14), the values of G_c in Table 6, computed by using Eq. (8) for the midpoint load test and for the specimens made of the same facing materials on both sides were used. According to Hoff and Mautner, when pa/2 is greater than 100 Eq. (11) which is essentially Eq. (8) holds. However, looking at the computed values of pa/2 in Table 8 all the numbers except those for the specimens, FGUCFG(R) are far below 100.

This implies that the real values of D for these specimens are less computed by Eq. (2), and thus the real values of G_c are greater than those in Table 6. The possibility of a decrease

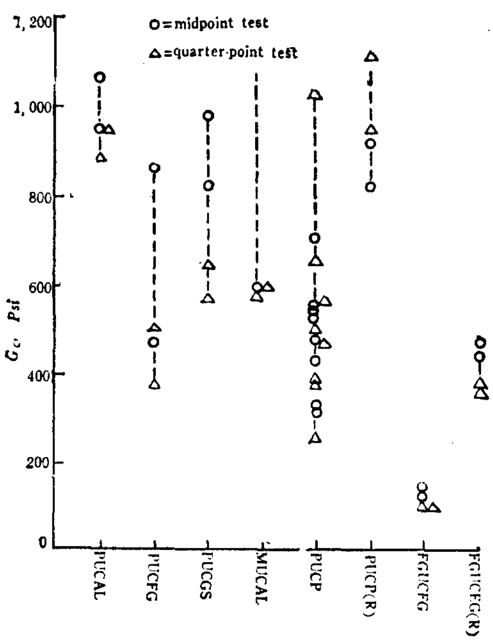

Fig. 7. Modulus of Rigidity of Core for Samples with the Core of Urecomb

in D may be supported by two statements as follows: First, in the analysis of Hoff and Mautner, when pa/2 is less than 0.1 Eq. (10) holds. Second, according to Singh, et al. (11), for a sandwich having a facing of considerable individual stiffness two neutral axes exist.

Table-5. pa/2[p is defind by Eq. (14)]

Sample name	pa/2
PSUP—1	8.06
PSUP—2	6.98
PSUP—3	6.17
PSUP—4	5.73
PSUP—5	4.24
PSUP—6	5.28
FGSUFG—1	43.11
FGSUFG—2	38.62
FGSUFG—3	46.86
FGSUFG—4	53.11
PPCP—1	6.94
PPCP—2	6.94
PUCP—1	9.18
PUCP—2	11.90
PUCP—3	11.67
PUCP—4	10.48
PUCP—5	8.97
PUCP—6	11.06
PUCP—7	15.85
PUCP—8	13.76
FGUCFG—1	46.04
FGU FG—2	44.39
FGUCFG(R)—1	112.62
FGUCFG(R)—1	117.14
PUCP(R)—1	13.14
PUCP(R)—2	14.02

The compressive facings stress σ_1 and the tensile facing stress σ_2 were calculated by Eq. (16) and Eq. (17). and the core shear stress 1 were calculated by Eq. (21). The stresses σ_1, σ_2 and were calculated by Eq. (19) and Eq. (22), which are simplieied equations for the panels with facings of the same material. These stresses were calculated based on the yield loads $P_{1\ yld}$ presented in Table 3.

The stresses σ_1 and σ_2 are, in most specimens, far below the reported strengths of the facing materials concerned. This indicates that yield or failure of the panel system occurs for causes other than tensile or compressive failure of the facing materials. In fact, failures in most of the specimens have taken place in the mode of either glue line slip, buckling on the top facing, or local wrinkling. It was found that fabrication of panels using a special rigid glue not only improves the effective G_c value of the core material, but also increases overall flexural strength by improving the shearing strength of the glue line.

It was shown that while most of the sample panels designated by PSUP and PUCP failed in the mode of glue line slip, thə panels designated by PUCP (R) failed in the mode of either shearing rupture of the core or in the mode of compressive rupture on the top facing.

V. Conclusions and Recommendations

The following conclusions and recommendations can be made based on the results on the experiment:

1. The deflection of the sandwich panel at the center can be approximately calculated by using analytically derived formulae, provided the elastic constants of the component materials are given.
2. Providing a good quality glue line between facing and core can improve the over-all flexural stiffness and flexural strength of a sandwhich panel by making the rigidity and strength of the component core material fully utillized.
3. The simplest forms of formulae for calculating maximum deflection and stresses exerted for the panels may be used for design purposes within the range of materials and sizes of the sandwhich panels used in this experiment.
4. Calculation of D and G_c by solving two different equations obtained from different modes of loading and by providing flexural test data for a sandwhich panel is not valid, in reality, because of the variation of the test data.
5. Further tests would be required to determine the relationship between slip at the glue line and horisontal shear stress. In order to account for this in the determination of the mechanical properties of the panels D and G_c other formulai which do not assume rigid glue lines must be used.
6. However, the data presented do indicate the relative stiffness and strength of the panels tested and do provide some guide lines for design if used conservatively.

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Appendix

VI. Matidiatical Development of Equations

The simply supported strip of sandwich construction is considered to be made up of two cantilever beams. The relations between stress and strain and the conditions of equilibrium and strain compatibility in the facings and core of the sandwich strip lead to a differential equation that is satisfied by a stress function. A suitable stress function is chosen and fitted to the proper boundary conditions of each facing and of the core. When this is done, it is found that only three constants remain to be determined by the conditions at the fixed and of the cantilever. These constants are determined by placing the horizontal displacements attop surface of the upper facing at the bottom surface of the lower facing and the vertical displacement near the center of the core(at the origin of the coordinate system used) equal to zero. Thus, the facings anid the core are not restrained from rotaing about their associated points of restraint except by their interactions with each other. The result is that their individual stiffness in bending are neglected at points directly under the central load. Therefore, the theory developed leads to a conservative estimate of flexural rigidity if the individual stiffness of the facings do contribute substantially to the total stiffness of the sandwich strip. Both core and faces will be assumed to be made of orthotropic materials, such as wood. The result can be extended immediately to cases where one or all of the materials are isotropic.

The thickness of the facings will be denoted by f_1 and f_2, respectively, that of the core by c, and the total thickness by h. The width of the strip will be denoted by b. The neutral plane,

$z=0$ in Figure 1, is taken to be at distance q from the facing whose thickness is f_1. The value of q will be determined in the course of the analysis. The difference, $c-q$, will be donoted by p.

The reduction in stiffness of a rectangular strip of length a, as shown in Figure 1, will be determined by assuming aload P to be at the center along a line perpendicular to the direction of the span. The strip will be considered to be made up of two cantilevers fixed at their junction $x=0$ and under the action of a load $P/2$ at the end of each, namely at $x=a/2$ and $x=a/2$. The width of the strip will be taken to be large in comparison with its thickness, so that the cantilever may be considered to be approximately in a state of plane strain. One of the cantilevers under consideration is shown in Fig. 8.

In the state of plane strain it is assumed that the components of displacement u and w parallel to the axes of x and z, respectively, are functions of x and z only, and that the component v parallel to the axis of y is zero. All components of stress and strain and strain are independent of y. The strain components e_{xy}, e_{yz}, and e_y, and stress components σ_{xy}, σ_{yx} all vanich. The stress component σ_{yy} is, in general,, not zero. Hence, to maintain the strip in state of plane

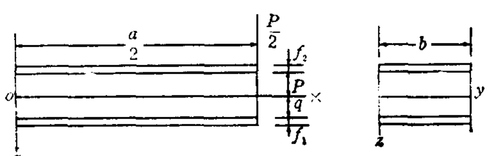

Fig 8. Half of a Simple Supported Panel as a Cantilever

strain, tensile on compressive forces must be applied on the faces $y=0$ and $y=b$ of the strip. The influence of these applied forces on the deflection of the cantilever is assumed to be negligible.

At the planes of separation between the facings and the core the following conditions hold:

The components of stress σ_{zz} and σ_{xz} are continuous.

The components of displacement d and w are continuous.

Within each layer the components of strain and stress are connected by the following relations, if the axes of x,y, and z are assumed to be normal to the planes of symmetry of the orthotrophic materials of the faces and core.

$$\left.\begin{aligned} e_{xx}&=\frac{1}{E_x}\sigma_{xx}-\frac{u_{yx}}{E_y}\sigma_{yy}-\frac{u_{zx}}{E_z}\sigma_{zz} \\ e_{yy}&=-\frac{u_{xy}}{E_x}\sigma_{xx}+\frac{1}{E_y}\sigma_{yy}-\frac{u_{zy}}{E_z}\sigma_{zz} \\ e_{zz}&=-\frac{u_{xy}}{E_x}\sigma_{xx}-\frac{u_{yz}}{E_y}\sigma_{yy}+\frac{1}{E_z}\sigma_{zz} \end{aligned}\right\} \quad \text{(A1)}$$

$$e_{xz}=\frac{1}{G_{xz}}\sigma_{xz} \quad \text{(A2)}$$

In these equations E_x, E_y, and E_z are Young's moduli in the directions x, y, and z, respectively. Poisson's ratio u_{xy} is the ratio of the contraction parrel to the y-axis to the extension parallel to the x-axis associated with a tension parallel to the x-direction. The quantity G_{xz} is the modulus of rigidity associated with the directions x and z,

In the respective layers the components of stress and strain and the contants of the materials will be denoted by subscripts 1, 2, and c. The subscript 1 will refer to the facing of thickness f_1, 2 to the facing of thickness f_2, and c to the core.

Since

$$e_{yy}=0 \quad \text{(A3)}$$

$$\sigma_{yy}=\frac{E_y}{E_x}u_{xy}\ \sigma_{xx}+\frac{E_y}{E_z}u_{zy}\sigma_{zz} \quad \text{(A4)}$$

Substituting (4) in (1), it is found that in each layer

$$e_{xx}=\frac{1}{E_x}(1-u_{xy}u_{yx})\sigma_{xx}-\frac{1}{E_z}(u_{yz}u_{zx}+u_{zx})\sigma_{zz}$$

(A5)

$$e_{zz}=-\frac{1}{E_x}(u_{xy}u_{yz}+u_{xz})\sigma_{xx}+\frac{1}{E_z}(1-u_{yz}u_{zy})\sigma_{zz}$$

Nothing that (p20(17))

$$u_{yx}=\frac{E_y}{E_x}u_{xy},\ u_{zy}=\frac{E_z}{E_y}u_{yz},\ \ u_{zx}=\frac{E_z}{E_x}u_{xz},$$

equations (A5) may be written

$$\left.\begin{aligned} e_{zz}&=\alpha\sigma_{xx}-\beta\sigma_{zz} \\ e_{zz}&=-\beta\sigma_{xx}+\gamma\sigma_{zz} \end{aligned}\right\} \quad \text{(A6)}$$

where

$$\left.\begin{aligned}\alpha&=\frac{1}{E_x}(1-\mu_{xy}\mu_{yx}),\\ \beta&=\frac{1}{E_x}(\mu_{xy}\mu_{yz}+\mu_{xz}),\\ \gamma&=\frac{1}{E_z}(1-\mu_{yz}\mu_{zy}).\end{aligned}\right\}\qquad (A7)$$

Within each layer of the sandwich the equations of equilibrium of the stress components σ_{xx}, σ_{xz}, and σ_{zz} assure the existence of a stress function F such that

$$\sigma_{xx}=\frac{\partial_2 F}{\partial_z^2},\quad \sigma_{zz}=\frac{\partial^2 F}{\partial_x^2},\quad \sigma_{xz}=\frac{\partial^2 F}{\partial_x\partial_z}\qquad (A8)$$

Substituting(A8) in (A6), and then making use of the compatibility equation

$$\frac{\partial^2 e_{xx}}{\partial_z^2}+\frac{\partial^2 e_{zz}}{\partial_x^2}=\frac{\partial^2 e_{xz}}{\partial_x\partial_z},$$

it follows that the stress function F satisfies the differential equation

$$\gamma\frac{\partial^4 F}{\partial_x^4}+\left(\frac{1}{G_x}-2\beta\right)\frac{\partial^4 F}{\partial_x^2\partial_z^2}+\alpha\frac{\partial^4 F}{\partial_z^4}=0\qquad (A9)$$

A suitable solution is

$$F=g(x-a/2)(z^3/3+ez)\qquad (A10)$$

Expressions of the form (A2), (A6), and (A10) hold for each layer separately. Equation (A10) will have the following forms in the core and facings 1 and 2, respectively.

$$\left.\begin{aligned}F_c&=g_c(x-a/2)(z^3/3+e_cz)\\ F_1&=g_1(x-a/2)\ (z^3/3+e_1z)\\ F_2&=g_2(x-a/2)(z^3/3+e_2z)\end{aligned}\right\}\qquad (A10a)$$

The constants that appear are to be determined by the conditions that hold on the planes separting the facings and the core, from the condition

$$\int_{-(p+f_2)}^{(q+f_1)}\sigma_{xz}dz=\frac{p}{2b}\qquad (A11)$$

and from the conditions that

$$\sigma_{xz}=0,\ \text{at}\ z=-(p+f_2)\ \text{and}\ z=(q+f_1)\qquad (A12)$$

It follows from (A8) and (A10a) that in the core

$$(\sigma_{xz})_c=-g_c(z^2+e_c)\qquad (A13)$$

$$(\sigma_{xx})_c=2g_c(x-a/2)z\qquad (A14)$$

$$(\sigma_{zz})_c=0\qquad (A15)$$

For the facings 1 and 2, the subscripts c are to be replaced by 1 and 2, respectively.

Equations(A2) and (A6), together with(A13), (A14), and (A15), give the following expressions for the compoents of strain in the core:

$$(e_{xx})_c=\frac{\partial u_c}{\partial_x}=2\alpha_c g_c(x-a/2)z\qquad (A16)$$

$$(e_{zz})_c=\frac{\partial w_c}{\partial z}=-2\beta_c g_c(x-a/2)z\qquad (A17)$$

$$(e_{xz})_c=\frac{\partial u_c}{\partial_z}+\frac{\partial w_c}{\partial_x}=\frac{1}{G_c}\sigma_{xz}=-\frac{g_c}{G_c}(z^2+e_c)\qquad (A18)$$

where u_c and w_c denote components of displacement in the core and G_c denotes the value of the modulus of rigidity G_{xz} in the core.

Again the corresponding equations for the facings are found by replacing the subscript c by 1 and 2, respectively. where $r_c(z)$ and $s_c(x)$

$$u_c=\alpha_c g_c(x-a/2)^2z+r_c(z)\qquad (A19)$$

$$w_c=-\beta_c g_c(x-a/2)z^2+s_c(x)\qquad (A20)$$

are arbitrary functions which are to be determined, apart from linear terms, by substitution of(A19) and(A20) in(A18). On substituting in (A19) and(A20) the functions determined in this way the following expressions for the components of the displacement in the core are obtained.

$$u_c=\alpha_c g_c\left(x-\frac{a}{2}\right)^2 z-\frac{g_c}{G_c}\left(\frac{z^3}{3}+e_cz\right)+\frac{\beta_c g_c}{3}z^3+k_cz+m_c\qquad (A21)$$

$$w_c=-\beta_c g_c\left(x-\frac{a}{2}\right)z^2-\frac{\alpha_c g_c}{3}\left(x-\frac{a}{2}\right)^3-k_cx+n_c\qquad (A22)$$

By writing the subscripts 1 and 2, respectively, in place of c, the corresponding expressions for the components of displacement in the facings are obtained.

The condition that the component of displacement u shall be continuous at the plane, $Z=q$, requires that

$$\alpha_c g_c\left(x-\frac{a}{2}\right)^2 q-\frac{g_c}{G_c}\left(\frac{q^3}{3}+e_cq\right)+\frac{\beta_c g_c}{3}q^3+k_cq+m_c=\alpha_1 g_1\left(x-\frac{a}{2}\right)^2 q-\frac{g_1}{G_1}\left(\frac{q^3}{3}+e_1q\right)+\frac{\beta_1 g_1}{3}q^3+k_1q+m_1\qquad (A23)$$

This relation is an identity in x, Hence

$$\alpha_c g_c=\alpha_1 g_1\qquad (A24)$$

and

$$-\frac{g_c}{G_c}\left(\frac{q^3}{3}+e_cq\right)+\frac{\beta_c g_c}{3}q^3+k_cq+m_c=-\frac{g_1}{G_1}\left(\frac{q^3}{3}+e_1q\right)+\frac{\beta_1 g_1}{3}q^3+k_1q+m_1\qquad (A25)$$

The continuity of the component w at the plane $z=q$ requires that

$$-\beta_c g_c\left(x-\frac{a}{2}\right)q^2-\frac{\alpha_c g_c}{3}\left(x-\frac{a}{2}\right)-k_c x+n_c$$
$$=-\beta_1 g_1\left(x-\frac{a}{2}\right)q^2-\frac{\alpha_1 g_1}{3}\left(x-\frac{a}{2}\right)^3-k_1 x+n_1 \tag{A26}$$

This identity in x yields the further relation

$$\beta_c g_c q^2+k_c=\beta_1 g_1 q^2+k_1 \tag{A27}$$

$$\beta_c g_c q^2\frac{a}{2}+n_c=\beta_1 g_1 q^2\frac{a}{2}+n_1 \tag{A28}$$

The following equations, corresponding to (A 24), (A25), (A27), and (A28), are obtained from the conditions that the components u and w are continuous at the plane $x=p$.

$$\alpha_c g_c=\alpha_2 g_2 \tag{A29}$$

$$\frac{g_c}{G_c}\left(\frac{p^3}{3}+e_c p\right)-\frac{\beta_c g_c}{3}p^3-k_c p+m_c$$
$$=\frac{g_2}{G_2}\left(\frac{p^3}{3}+e_2 p\right)-\frac{\beta_c g_c}{3}p^3-k_2 p+m_2 \tag{A30}$$

$$\beta_c g_c p^2+k_c=\beta_2 g_2 p^2+k_2 \tag{A31}$$

$$\beta_c g_c p^2\frac{a}{2}+n_c=\beta_2 g_2 p^2\frac{a}{2}+n_2 \tag{A32}$$

By comparing (24) and (29), it is seen that

$$\alpha_1 g_1=\alpha_2 g_2 \tag{A33}$$

It will be convenient to introduce the notation

$$\left.\begin{aligned}p_1&=\frac{\sigma_1}{\alpha_c}=\frac{(E_x)_c(1-\mu_{xy}\mu_{yx})_1}{(E_x)_1(1-\mu_{xy}\mu_{yx})_c}\\ p_2&=\frac{\alpha_1}{\alpha_c}=\frac{(E_x)_c(1-\mu_{xy}\mu_{yx})_1}{(E_x)_1(1-\mu_{xyy\,x})_c}\end{aligned}\right\} \tag{A34}$$

Then in accordance with (24) and (29)

$$\left.\begin{aligned}g_c&=\frac{\alpha_1}{\alpha_c}g_1=p_1 g_1\\ g_c&=\frac{\alpha_2}{\alpha_c}g_2=p_2 g_2\end{aligned}\right\} \tag{A35}$$

$$\beta_1\neq\beta_2 \tag{A36}$$

in general, if the facings are not made of the same material.

By introducing the following notations:

$$\frac{(E_x)_2}{(E_x)_1}=n_E,\quad \frac{\alpha_2}{\alpha_1}=n_\alpha,\quad \frac{\beta_2}{\beta_1}=n_\beta,$$
$$\frac{\lambda_2}{\lambda_1}=n_\lambda \quad\text{and}\quad \frac{G_2}{G_1}=n_g,$$

g_2 can be replaced by g_1/n_α β_2 by n_β β_1, α_2 by n_α α_1, $(E_x)_2$ by n_E E_1, G_2 by n_G, and g_c by $p_1 g_1$.

The condition that the component of shearing stress σ_{xz} is continuous at the planes $z=q$ and $z=-p$ requires that

$$p_1 g_1(q^2+e_c)=g_1(q^2+e_1) \tag{A37}$$

$$p_2 g_2(p^2=e_c)=g_2(p^2=e_2) \tag{A38}$$

Further, it follows from (A12) that

$$(q+f_1)^2+e_1=0 \tag{A39}$$

$$(p+f_2)^2+e_2=0 \tag{A40}$$

Hence $e_1=-(q+f_1)^2$ (A41)

$$e_2=-(p+f_2)^2 \tag{A42}$$

On substituting (41) and(42) in(37) and (38), respectively, it is found that

$$e_c=-q^2-\frac{1}{p_1}(2qf_1+f_1^2) \tag{A43}$$

$$e_c=-\ p^2-\frac{1}{p_2}(2pf_2+f_2^2) \tag{A44}$$

It is clear that q, the distance from the neutral plane $z=0$ to the Junction of the core and facing f_1, must be chosen so that the two expressions for e_c are equal.

By equating these expressions and recalling that

$p=e-q$, itis found that

$$q=\frac{(1/n_p)f_2^2-f_1^2+2(1/n_p)cf_2+p_1c^2}{2[f_1+(1/n_p)f_2+p_1{}^c]} \tag{A45}$$

To complete the determinations of the constants that appear in the expressions for u_c, w_c, u_1, w_1, u_2, and w_2, the following conditions are imposed at the fixed end $x=0$ of the cantilever forming the right-hand half of the beam.

$w_c=0$	$x=0$	$z=0$	(A46)
$u_1=0$	$x=0$	$z=q+f_1$	(A47)
$u_2=0$	$x=0$	$c=-(p+f_2)$	(A48)

Similar boundary conditions were found to lead to satisfactory conclusions in the case of a plywood strip (5).

From conditions (A47) and (A48) and equation (A21) written with subscripts 1 and 2, respectively, and using (A41) and (A42), the following are obtained:

$$\alpha_1 g_1\frac{a}{4}(q+f_1)+\frac{2}{3}\frac{1}{G_1}(q+f_1)^3+\frac{\beta_1 g_1}{3}(q+f_1)^3$$
$$+k_1(q+f_1)+m_1=0 \tag{A49}$$

$$-\alpha_1 g_1\frac{a^2}{4}(p+f_2)-\frac{2}{3}\frac{1}{n_G n_\alpha}\frac{g_1}{G_1}(p+f_2)^3$$
$$-\frac{n_\beta}{n_\alpha}\frac{\beta_1 g_1}{3}(p+f_2)^3-k_2(p+f_2)+m_2=0 \tag{A50}$$

From(A46) and(A22) it is found that:

$$n_c = -\frac{\alpha_c p_1 g_1 a^3}{24} \quad \text{(A51)}$$

Substitute k_c in terms of k_1 from(A27) in(A25) and k_c in terms of k_2 from(A31) in(A30) and substract, obtaining:

$$m_1 - m_2 = \frac{g_1}{G_1}\left(\frac{q^3}{3} + e_1 q + \frac{p^3}{3n_\alpha n_G} + \frac{c_2 p}{n_\alpha n_G}\right) - \frac{p_1 g_1}{G_c}\left(\frac{q^3}{3} + e_c q + \frac{p^3}{3} + e_c p\right) + \frac{2}{3}\beta_1 g_1\left(q^3 + \frac{n_\beta}{n_\alpha} p^3\right) - \frac{2}{3} p_1 \beta_c g_1 (q^3 + p^3) \quad \text{(A52)}$$

From (A27) and (A31)

$$k_2 = k_1 + \beta_1 g_2\left(q^2 - \frac{n_\beta}{n_\alpha} p^2\right) - p_1 \beta_c g_1 (q^2 - p^2) \quad \text{(A53)}$$

Substract(A50) from(A49) after substituting (A53) for k_2 in (A50) and obtain after some reduction:

$$m_1 - m_2 = -\alpha_1 g_1 \frac{a^2}{4} h - \frac{2}{3}\frac{g_1}{G_1}\left[(q+f_1)^3 + \frac{(p+f_2)^3}{n_\alpha n_G}\right] - \frac{\beta_1 g_1}{3}(q+f_1)^3 + \frac{n_\beta}{n_\alpha}(p+f_2)^3 - k_1 h - \beta_1 g_1 (p+f_2)\left(q^2 - \frac{n_\beta}{n_\alpha} p^2\right) + p_1 \beta_c g_1 (p+f_2)(q^2 - p^2) \quad \text{(A54)}$$

where: $h = q + f_1 + p + f_2$

Equate expressions for $m_1 - m_2$ in(A52) and(A54) and solve for k_1 and obtain after considerable reduction:

$$k_1 = -g_1\left\{\frac{\alpha_1 a^2}{4} + \frac{1}{G_1 h}\left[q f_1^2 + \frac{2}{3} f_1^3 + \frac{1}{n_\alpha n_G}\left(p f_2^2 + \frac{2}{3} f_2^3\right)\right] + \frac{\beta_1}{h}\left[q^2 h + q f_1^2 + \frac{f_1^3}{3} + \frac{n_\beta}{n_\alpha}\left(p f_2^2 + \frac{f_2^3}{3}\right)\right] - \frac{p_1 \beta_c}{h}\left[\frac{2}{3} q^3 + q^2 p - \frac{1}{3} p^3 + (q^2 - p^2) f_2\right] - \frac{p_1}{G_c h}\left(\frac{q^3}{3} + \frac{p^3}{3} e_c c\right)\right\} \quad \text{(A55)}$$

To obtain the deflection at the center of the beam the displacement w_1 at the end $x = a/2$, of the cantilever will be calculated. This will be measur edra with reference to a point on the plane of the neutral axis at the middle of the beam. Consequently, the deflection of points on the neutral plane at the center of the beam will be numerically equal to the quantity w_1 calculated at the end $x = a/2$ of the cantiever.

In accordance with(A22)

$$(w_1)_x = \frac{a}{2} = -k_1 \frac{a}{2} + n_1 \quad \text{(A56)}$$

From(51) and(28)

$$n_1 = g_1\left[p_1 \beta_c \frac{a}{2} q^2 - \beta_1 \frac{a}{2} q^2 - \frac{\alpha_1 a^3}{24}\right] \quad \text{(A57)}$$

On substituting(A55) and(A57) in (A56) the following expression is obtained after some reduction:

$$(w_1)_{x=a}/2 = g_1\left\{\frac{\alpha_1 a^3}{12} + \frac{a}{2G_1 h}\left[q f_1^2 + \frac{2}{3} f_1^3 + \frac{1}{n_\alpha n_G}\left(p f_2^2 + \frac{2}{3} f_2^3\right)\right] + \frac{\beta_1 a}{2h}\left[q f_1^2 + \frac{f_1^3}{3} + \frac{n_\beta}{n_\alpha}\left(p f_2^2 + \frac{f_2^3}{3}\right)\right] + \frac{p_1 B_c a}{2h}\left[\frac{q^3}{3} + \frac{p^3}{3} q^2 f_1 + p^2 f_2\right] - \frac{p_1 a}{2G_c h}\left[\frac{q^3}{3} + \frac{p^3}{3} + e_c c\right]\right\} \quad \text{(A58)}$$

and this expression can be further reduced to the form:

$$(w_1)_{x=a}/2 = \frac{g_1 \alpha_1 a^3}{12}\left\{1 + \frac{2}{a^2 h}\left[\frac{1}{\alpha_1 G_1}\left(3q f_1^2 + \frac{3}{n_\alpha n_G} p f_2^2 + 2 f_1^3 + \frac{2}{n_\alpha n_G} f_2^3\right) + \frac{\beta_1}{\alpha_1}\left(3q f_1^2 + \frac{3n_\beta}{n_\alpha} p f_2^2 + f_1^3 + \frac{n_\beta}{n_\alpha} f_2^3\right) + \frac{p_1 \beta_c}{\alpha_1}(q^3 + p^3 + 3q^2 f_1 + 3p^2 f_2) - \frac{p_1}{\alpha_1 G_c}(q^3 + p^3 + 3e_c c)\right]\right\} \quad \text{(A59)}$$

The coefficient g_1 can be calculated from the condition (A11).

$$\int_{-(p+f_2)}^{q+f_1} \sigma_{xz} d_z = -g_1/n_\alpha \int_{-(p+f_2)}^{-p} (z^2 + e_2) dz - p_1 g_1 \int_{-p_1}^{q} (z^2 + e_c) dz - g_1 \int_{q}^{(q+f_1)} (z^2 + e_1) dz$$

After performing the integrations and making use of (A41), (A42), (A43), and (A44) the right hand side of this equation reduced to:

$$\frac{2}{3} g_1\left[3q^2 f_1 + 3q f_1^2 + f_1^3 + \frac{1}{n_\alpha}(3p^2 f_2 + 3p f_2^2)\right.$$

$+p_1(q^3+p^3)$

By(A11) this expression is equal to $p/2b$. Hence:

$$g_1=\frac{3p}{4b[3q^2f_1+3qf_1^2+f_1^3+\frac{1}{n_\alpha}(3p^2f_2+3pf_2^2+f_2^2+f_2^3+p_1(q^3+p^3)]} \quad \text{(A60)}$$

The denominator is closely related to D, the stiffness of the beam as calculated without correcting for the effect of shear deformation. For,

$$D=b\int_{-(p+f_2)}^{-p}\frac{(E_x)_2z^2}{\lambda_2}dz+b\int_{-p}^{q}\frac{(E_x)_cz^2}{\lambda_c}dz+b\int_{q}^{q+f_1}\frac{(E_x)_1z^2}{\lambda_1}dz$$

where
$$\lambda_1=(1-u_{xy}u_{yx})_1$$
$$\lambda_2=(1-u_{xy}u_{yx})_2 \quad \text{(A60a)}$$
$$\lambda_c=(1-u_{xy}u_{yx})_c$$

After nothing that

$$\frac{(E_x)_2}{\lambda_2}=\frac{n_E(E_x)_1}{n_\lambda\lambda_1}=\frac{1}{n_\alpha\alpha_1} \text{ and } \frac{(E_x)_c}{\lambda_c}=\frac{1}{\alpha_c}$$

the expression for D is roadily reduced to:

$$D=\frac{3b}{\alpha_1}\Big[3q^2f_1+3qf_1^2+f_1^3+\frac{1}{n_\alpha}3p^2f_2+3pf_2^2+f_2^3)+p_1(q^3+p^3) \quad \text{(A61)}$$

where in accordance with (A34):

$p_1=\alpha_1/a_c$

It follows from (A60) and(A61) that:

$$g_1=\frac{P}{4\alpha_1 D} \quad \text{(A622)}$$

by using(A62), equation (A59) can be written in the form:

$$(w_1)_{x=a}/2=\frac{Pa^3}{48D}\Big(1+n\frac{h^2}{a^2}\Big) \quad \text{(A63)}$$

where:

$$n=\frac{2}{h^3}\Big[\frac{1}{\alpha_1G_1}\Big(3qf_1^2+\frac{3}{n_\alpha n_G}pf_2^2+2f_1^3+\frac{2}{n_\alpha n_G}f_2^3\Big)+\frac{\beta_1}{\alpha_1}\Big(3qf_1^2+\frac{3n_\beta}{n_\alpha}pf_2^2+f_1^3+\frac{n_\beta}{n_\alpha}f_2^3\Big)+\frac{p_1\beta_c}{1}(q^3+p^3+3q^2f_1+3p^2f_2)-\frac{p_1}{\alpha_1G_c}(q^3+p^3+3e_cc)\Big] \quad \text{(A64)}$$

In this expression q and e_c are to be calculated by formulas(A45) and(A43). Further $p=c-q$.

As will be seen from the steps taken to calculated it, the stiffness D is the stiffness that would be determined in a load-deflection test of a centrally loaded beam if a correction for shear deformation were not necessary. Equation(A63) shows that the effective stiffness of a centrally loaded strip of sandwich is equal to D divided by $1+n\ h^2/a^2$. Consequently,

$$\text{Effective stiffness}=\frac{D}{1+n\frac{h^2}{a^2}} \quad \text{(A65)}$$