THE DESIGN OF AN OPTIMAL SPARE KIT FOR WEAPON SYSTEMS

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Abstract

One of the pending issues of the Ministry of Defense is the efficient management of space parts for the weapon systems. It has been known that more than 000 million dollars are needed for spare parts for the weapon systems annually. Though the problem demands a serious consideration, there has not been any systematic study on the problem as far as the author knows. One way to approach the problem is through an investigation of the system reliability under constraints.

A measure of how well a system performs or meets its design objectives is provided by the concept of system reliability. If successful operation is desired for a specified period of time, reliability is defined as the probability that the system will perform satisfactorily for the required time period. This interpretation of reliability is normally applied to devices which are subject to random failures such as electrical or mechanical systems.

It has been found necessary to express system reliability in terms of the reliability of the components or subsystems which comprise the system. The major subsystems of an aircraft, for example, include the electronics, powerplant, airframe and armament subsystems. Therefore, the optimal spare part kit can be found by maximizing the system reliability subject to cost or other constraints.

1. The Reliability Function

When a fixed number N of identical systems are repeatedly operated, there will be, after a time t, $N_s(t)$ test and $N_f(t)$ that fail. The reliability of such a system can then be expressed as the following ratio:

$$R(t)=N_s(t)/N=1-N_f(t)/N \qquad (1)$$

Letting N be fixed, we have

$$dR(t)/dt=(-1/N)\ dN_f(t)/dt$$

or

$$dN(t)/dt=-N\ dR(t)/dt \qquad (2)$$

Dividing both sides of equation (2) by N_s, we obtain the instantaneous probability of failure, denoted by $r(t)$, i.e.,

$$r(t)=[1/N_s(t)]\ dN_f(t)/dt$$
$$=[-N/N_s(t)]dR(t)/dt \qquad (3)$$

Since $R(t)=N_s(t)/N$, we have

$$r(t)=[-1/R(t)]\ [dR(t)/dt]$$

Therefore,

$$R(t)=\exp(-\int_0^t r(x)\ dx) \qquad (4)$$

The function $r(t)$ is called the failure rate function.

If the system has a failure time density function $f(t)$, then the failure time distribution function is given by

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$$F(t)=\int_0^t f(x)dx \qquad (5)$$

called the unreliability of the system at time t. Thus, the system reliability can be given by

$$R(t)=1-F(t)=\exp(-\int_0^t r(x)dx) \qquad (6)$$

2. Basic Sparing Criteria

One basic criterion for selection of the number of spares needed is to determine the number of spares that should be provided in order to assure with remain in operation for a specified length of time. Let $f_k(t)$ denote the density function for the time to the k_{th} failure. If we allocate N spare devices, then the probability that this number of spares will be adequate for an interval of length t is merely the probability that there will be no more than N failures in $[0, t]$; hence

$$\Pr\{N \text{ spares are adequate for } [0, t]\}$$
$$=\Pr\{(N+1)\text{st failure time} \geq t\}$$
$$=\int_t^{+\infty} f_{N+1}(x)dx \qquad (7)$$

If the device has constant failure rate λ, then the sum of k independent, identical, exponentially distributed random variables is distributed by the Gamma density function

$$f_k(t)=\frac{(\lambda t)^{k-1}e^{-\lambda t}}{(k-1)!} \qquad (8)$$

Thus by integrating by parts, we obtain

$$\int_t^{+\infty} f_{N+1}(x)dx=\sum_{j=0}^{N}\frac{(\lambda t)^j e^{-\lambda t}}{j!} \qquad (9)$$

The quantity of the right side is the sum of the Poisson distribution function. To determine the smallest value of N for which

$$\sum_{j=0}^{N}\frac{(\lambda t)^j e^{-\lambda t}}{j!} \geq \xi \qquad (10)$$

for a given probability ξ, thus requires that one refer to a table the Poisson distribution function.

When maximum system reliability, however, is the criterion for the spare parts kit, the kit can be selected as follows. Suppose that the system is required to operate during $[0,t]$ and the system consists of n distinct components arranged in series as in Fig.1.

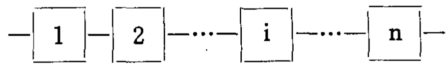

Fig. 1

When a component fails, it is instantly replaced by a spare if one is available. The components are considered to operate independently of each other and are essential to continued system operation. Assume that only the spare parts provided in a kit, say m_i-1 for component i, is used for replacement. Let $P_i(m_i)$ be the probability that m_i or fewer failures of type i occur in $[0, t]$ for $i=1, 2, \ldots, n$. Then the probability of system operation without failure for the interval $[0, t]$ is given by

$$P(m_1, m_2, \ldots, m_n)=\prod_{i=1}^{n} P_i(m_i) \qquad (11)$$

The problem is to choose the vector$(m_1, m_2, \ldots m_n)$ so as to maximize $P(m_1, m_2, \ldots, m_n)$. We will consider this problem when there exist three constraints of the forms

$$\sum_{i=1}^{n} c_i m_i \leq C \qquad (12)$$

$$\sum_{i=1}^{n} w_i m_i \leq W \qquad (13)$$

$$m_i=\text{integers} \qquad (14)$$

3. The Optimal Allocation of Spare Parts

Suppose that the components of type i in a system under consideration have constant failure rate λ_i, then the reliability of m_i spares for the i_{th} component is given by

$$P_i(m_i)=\sum_{j=1}^{m_i}\frac{(\lambda_i t)^j e^{-\lambda_i t}}{j!} \qquad (15)$$

Since $m_1, m_2, \ldots, m_n$ can take on only integers the above problem can be formulated as a dynamic programming problem with three constraints:

$$\text{Maximzie } R_n = \sum_{i=1}^{n} P_i(m_i)$$

subject to

$$m_i = 0, 1, 2, \cdots \text{integers}$$

$$\sum_{i=1}^{n} c_i m_i \leq C \text{ (cost constraint)} \qquad (16)$$

$$\sum_{i=1}^{n} w_i m_i \leq W \text{ (weight constraint)}$$

If we formulate this problem as a conventional two constraints dynamic programming problem of dealing with sequences of functions of two variables with the following general recurrence relation of the form

$$f_n(C,W) = \underset{0\leq m_n\leq \min[C/c_n,\, W/w_n]}{\text{Max}} \sum_{j=1}^{m_n} \frac{(\lambda_n t)^j e^{-\lambda_n t}}{j!} \cdot f_{n-1}(C - m_n c_n,\ W - m_n w_n) \qquad (17)$$

This formulation involves a great deal of memory capacity in computationand, consequently, will require more computing time. Therefore, we'd better follow the method suggested by R.E. Bellman i.e., the introduction of Lagrange multiplier. Consider the new problem of maximizing the expression

$$\prod_{i=1}^{n} P_i(m_i)\ e^{-\alpha \sum_{i=1}^{n} m_i w_i}$$

subject to

$$m_i = 0, 1, 2, \ldots \text{integers} \qquad (18)$$

$$\sum_{i=1}^{n} c_i m_i \leq C$$

over all m_i satisfying only the first two constraints of the original problem. Setting $f_n(C)$ equal to this maximum value, we have the recurrence relation

$$f_n(C) = \underset{0\leq m_n\leq [C/c_n]}{\text{Max}} \sum_{j=0}^{m_n} \frac{(\lambda_n t)^j e^{-\lambda_n t}}{j!} e^{-\alpha m_n w_n \cdot f_{n-1}(C - m_n c_n)} \qquad (19)$$

Thus, for $n=1$ we have

$$f_1(C) = \underset{0\leq m_1\leq [C/c_1]}{\text{Max}} \sum_{j=0}^{m_1} \frac{(\lambda_n t)^j e^{-\lambda_n t}}{j!} e^{-\alpha_1 m_1 w_1} \qquad (20)$$

Suppose that the failure rates, unit cost and unit weight are given as follows:

Note that we do not have to consider f_1 for all values of C: since the unit cost of component 1 is 4.0 we need to compute f_1 for 4.0, 8.0, 12.0,... of C. Furthermore, it is clear that we have at least one component for all stages: otherwise the reliability will be zero. Thus, it is unnecessary to consider values of C greater than $C-(c_2+c_3)$, i.e., 12 units of C. This becomes clear if the recurrence relation for $n=2$ is examined.

The recurrence relation for $n=2$ is as follows:

$$f_2(C) = \underset{0\leq m_2\leq [C/c_2]}{\text{Max}} \sum_{j=0}^{m_2} \frac{(\lambda_2 t)^j e^{-\lambda_2 t}}{j!} e^{-\alpha_1 m_2 w_2 \cdot f_1(C - m_2 c_2)}$$

$$= \underset{0\leq m_2\leq [C/c_2]}{\text{Max}} \sum_{j=0}^{m_2} \frac{(0.3)^j e^{-0.3}}{j!} e^{-0.04 m_2 \cdot f_1(C - 2.0 m_2)} \qquad (21)$$

since we take the time interval equal to a unit period. The result of computations for the above relation is shown in Table 3.

Finally, for $n=3$ we have the recurrence relation of the form

$$f_3(C) = \underset{0\leq m_3\leq [C/c_3]}{\text{Max}} \sum_{j=0}^{m_3} \frac{(0.1)^j e^{-0.1t}}{j!} e^{-0.08 m_3 \cdot f_2(C - 6.0 m_3)} \qquad (22)$$

Table 4 shows the result of the computat-ions for f_3.

TABLE 1 FAILURE RATE, UNIT COST AND UNIT WEIGHT

Component	Failure rate	Unit cost	Unit weight
1	0.2	4.0	5.0
2	0.3	2.0	4.0
3	0.1	6.0	8.0

Furthermore, let's assume that the cost allowable is 20 units and the allowed maximum weight is 30 units. Since there is no way to guess the exact value of α_1 in applying the formulation (18), set $\alpha_1=0.01$ an arbitrary value. If we proceed with this α_1 we will find the following values for the computation of equation (20):

TABLE 2 COMPUTATION OF f_1

C	m_1	$P_1(m_1)$	$e^{-\alpha_1 m_1 w_1}$	$P_1(m_1)e^{-\alpha_1 m_1 w_1}=f_1$
4	1	.8187	.9512	.7788
8	2	.9825	.9048	.8890
12	3	.9989	.8607	.8598

TABLE 3 COMPUTATION OF f_2

C	m_2	$\sum_{j=0}^{m_2}\frac{(0.3)^j e^{-0.3}}{j!}e^{-0.04m_2}]$ (A)	$m_1=\frac{C-2.0m_2}{4}$	f_1	Af_1	f_2
8	2	.9631	1	.7788	.6924	.6924
	1	.7118	1	.7788	.5544	
14	5	.8187	1	.7788	.6703	
	4	.8519	1	//	.6635	
	3	.8837	2	.8890	.7856	
	2	.8891	2	//	.7904	.7904
	1	.7118	3	.8598	.6120	

TABLE 4 COMPUTATION OF f_3

C	m_3	$\sum_{j=0}^{m_3}\frac{(0.1)^j e^{-0.1}}{j!}e^{-0.83m_3}]$ (A)	$C-6.0m_3$	f_2	Af_2	f_3
20	2	.8481	8	.6924	.5873	
	1	.8352	14	.7904	.6601	.6601

Therefore, for $\alpha_1=0.01$ the optimal values are $m_1=2$, $m_2=2$ and $m_3=1$. However, as the sum $W_1=\sum_{i=1}^{n} m_i w_i=26<30$ shows the weight limitation is not yet effective. Thus, we may decrease the trial value for α to, say $\alpha_2=0.006$. If the sum $W_2=\sum_{i=1}^{n} m_i w_i$ does not approach 30 for the new α_2, then we may find another value α_3 either through interpolation or extrapolation. One simple way is to compute

$$\alpha_3=\frac{\alpha_2-\alpha_1}{W_2-W_1}(W-W_1)+\alpha_1$$

knowing α_2, α_1, W_1 and W_2.

Now we repeat the computations from f_1 to f_3 for $\alpha_2=0.0006$. For $n=1$ we have the recurrence relation

$$f_1(C)=\max_{0\leq m_1\leq[C/c_1]}\sum_{j=1}^{m_1}\frac{(0.2)^j e^{-0.2}}{j!}\cdot e^{-0.03m_1} \quad (23)$$

Table 5 shows the result of computations.

TABLE 5 COMPUTATION OF f_1

C	m_1	$\sum_{j=0}\frac{(0.2)^j e^{-0.2}}{j!}\cdot e^{-0.03m_1}$
4	1	.7945
8	2	.9253
12	3	.9129

The second stage recurrence relation is given as follows:

$$f_2(C)=\max_{0\leq m_2\leq[C/c_2]}\sum_{j=0}^{m_2}\frac{(0.3)^j e^{-0.3}}{j!}\cdot e^{-0.03m_2}\cdot f_1(C-2.0m_2) \quad (24)$$

And the third stage becomes

$$f_3(C)=\max_{0\leq m_3\leq[C/c_3]}\sum_{j=0}^{m_3}\frac{(0.1)^j e^{-0.1}}{j!}\cdot e^{-0.048m_3}\cdot f_2(C-6.0m_3) \quad (25)$$

Table 6 and 7 show the results of computations for f_2 and f_3.

TABLE 6 COMPUTATION OF f_2

C	m_2	$\sum_{j=0}^{m_2}\frac{(0.3)^j e^{-0.03m_2}}{j!}\cdot e^{-0.03m_2}$ (A)	$C-2.0m_2$	f_1	Af_1	f_2
8	2	.9070	4	.7788	.7064	.7064
	1	.7189	6	//	.5599	
12	5	.8607	4	//	.6703	
	4	.8867	6	//	.6903	
	3	.9106	8	.8890	.8096	.8096
	2	.9070	10	//	,8063	
	1	.7198	12	.8598	.6181	·

TABLE 7 COMPUTATON OF f_3

C	m_3	$\sum_{j=0}^{m_3}\left[\frac{(0.1)^j e^{-0.1}}{j!}\right]^{-0.04m_3}$ (A)	$C-6.0m_3$	f_2	Af_2	f_3
20	2	.9042	8	.7064	.6388	
	1	.8624	14	.8096	.6982	.6982

Note that in Table 6 we do not have consider all values of C since the third stage recurrence relation leave cost allowance 14, 8 and 2 for the first two stages. Furthermore, we can easily see that $C-6.0m_3$ corresponding to the value of $m_3=3$ is not optimal since there is no money left for at least one of m_1. Thus, when $\alpha_2=0.006$ the optimal values for m_i's are

$$m_1=2 \quad m_2=3 \quad m_3=1$$

Moreover, $\sum_{i=1}^{3} m_i w_i=30$. Thus the optimal values are global optimal values of the original problem (16). The optimal spare kit comprises one spare for the component, two spares for the second component and none for the third.

4. Conclusion

In the above discussions we have not investigated the inventory management problem. The problem we have analyzed is the optimal mix of spare parts initially provided to military units equipped with new weapon systems. This model can be applied to small arms as well as to complex weapon systems. Furthermore, the result can be easily expanded to analyze different configurations of subsystems. The so called stand-by system, for example, poses a much more simpler problem.

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