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ON CERTAIN CHAIN FORMULAE IN TWO DIMENSIONAL LAPLACE TRANSFORM II

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1. Introduction

The well known Laplace transform of a function $f(x,y)$ is given by Humbert [4] as:

$$F(p,q)=pq\int_0^\infty\int_0^\infty e^{-px-qy} f(x,y)\,dx\,dy,\ \mathrm{Re}(p,q)>0. \tag{1.1}$$

Symbolically, (1.1) is denoted as

$$f(x,y) \doteqdot F(p,q).$$

This paper is in continuation of my paper [6], where I have proved two theorems in transform (1.1). In this paper, further I have proved four theorems and many new results have been obtained by the application of these theorems.

2. THEOREM 1.

Let

$$f(x,y) \doteqdot \phi_1(p,q), \tag{2.1}$$

$$\frac{1}{\sqrt{xy}}\,\phi_1\left(\frac{1}{x},\frac{1}{y}\right) \doteqdot \phi_2(p,q), \tag{2.2}$$

then

$$f(x^2,y^2) \doteqdot \frac{4}{\pi pq}\,\phi_2\left(\frac{p^2}{4},\frac{q^2}{4}\right), \tag{2.3}$$

provided $\frac{1}{\sqrt{xy}}f(x,y)$ *and* $f(x,y)$ *are absolutely integrable in* $0\le x<\infty$, $0\le y<\infty$, $\mathrm{Re}(p,q)>0$.

PROOF. Let $\phi_1(p,q)=pq\int_0^\infty\int_0^\infty e^{-pu-qv} f(u,v)\,du\,dv.$

$$\therefore\ \frac{1}{\sqrt{xy}}\,\phi_1\left(\frac{1}{x},\frac{1}{y}\right)=(xy)^{-\frac{3}{2}}\int_0^\infty\int_0^\infty e^{\left(-\frac{u}{x}-\frac{v}{y}\right)} f(u,v)\,du\,dv.$$

From (2.2), we obtain

$$\phi_2(p,q)=pq\int_0^\infty\int_0^\infty e^{-px-qy}(xy)^{-\frac{3}{2}}\,dx\,dy\int_0^\infty\int_0^\infty e^{\left(-\frac{u}{x}-\frac{v}{y}\right)} f(u,v)\,du\,dv.$$

Changing the order of integrations, which is justified [1, p. 503] under the conditions mentioned in the theorem and evaluating the inner integral thus obtained we get the desired result on simplification.

APPLICATIONS.

1. Let $$f(x,y)=x^{c_1}\, y^{c_2}\, G^{rs}_{uv}\left(cxy\,\middle|\,\begin{matrix}a_1,\cdots,a_u\\ b_1,\cdots,b_v\end{matrix}\right).$$

$$\phi_1(p,q)=p^{-c_1}q^{-c_2}\, G^{r,s+2}_{u+2,v}\left(\frac{c}{pq}\,\middle|\,\begin{matrix}-c_1,-c_2,a_1,\cdots,a_u\\ b_1,\cdots\cdots\cdots,b_v\end{matrix}\right),\quad [5]$$

provided $u+v<2(r+s)$, $|\arg c|<\left(r+s-\frac{u}{2}-\frac{v}{2}\right)\pi$, $\mathrm{Re}(c_k+b_j+1)>0$, $k=1,2\,;\,j=1,2,\cdots,r$.

$$\phi_2(p,q)=p^{-c_1+\frac{1}{2}}\,q^{-c_2+\frac{1}{2}}\,G^{r,s+4}_{u+4,v}\left(\frac{c}{pq}\,\middle|\,\begin{matrix}-c_1,-c_1+\frac{1}{2},-c_2,-c_2+\frac{1}{2},a_1,\cdots,a_u\\ b_1,\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,b_v\end{matrix}\right),$$

provided $u+v<2(r+s)$, $|\arg c|<\left(r+s-\frac{u}{2}-\frac{v}{2}\right)\pi$, $\mathrm{Re}\left(c_k+b_j+\frac{1}{2}\right)>0$, $k=1,2\,;\,j=1,2,\cdots,r$.

$$\therefore\ x^{2c_1}\,y^{2c_2}\,G^{rs}_{uv}\left(cx^2y^2\,\middle|\,\begin{matrix}a_1,\cdots,a_u\\ b_1,\cdots,b_v\end{matrix}\right)\doteqdot\frac{2^{2(c_1+c_2)}}{\pi p^{2c_1}\,q^{2c_2}}$$
$$\times\, G^{r,s+4}_{u+4,v}\left(\frac{2^4c}{p^2q^2}\,\middle|\,\begin{matrix}-c_1,-c_1+\frac{1}{2},-c_2,-c_2+\frac{1}{2},a_1,\cdots,a_u\\ b_1,\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,b_v\end{matrix}\right),$$

provided $u+v<2(r+s)$, $|\arg c|<\left(r+s-\frac{u}{2}-\frac{v}{2}\right)\pi$, $\mathrm{Re}\left(c_k+b_j+\frac{1}{2}\right)>0$, $k=1,2\,;\,j=1,2,\cdots,r$, $\mathrm{Re}(p,q)>0$.

2. Let

$$f(x,y)=\begin{cases}J_0(2\sqrt{x}), & y>x\\ J_0(2\sqrt{y}), & y<x.\end{cases}$$

$$\phi_1(p,q)=e^{-\left(\frac{1}{p+q}\right)},\quad [2, p. 142]$$

$$\phi_2(p,q)=\frac{\pi\sqrt{pq}\,(\sqrt{p}+\sqrt{q})}{\sqrt{\{1+(\sqrt{p}+\sqrt{q})^2\}}},\quad [2, p. 139]$$

$$\therefore\ \frac{p+q}{\sqrt{\{4+(p+q)^2\}}}\doteqdot\begin{cases}J_0(2x) & \text{for } y>x\\ J_0(2y) & \text{for } y<x,\end{cases}\quad \mathrm{Re}(p,q)>0.$$

3. Let

$$f(x,y)=\begin{cases}\dfrac{x^a}{\Gamma(a+1)} & \text{for } y>x\\[2mm] \dfrac{y^a}{\Gamma(a+1)} & \text{for } y<x.\end{cases}$$

$$\phi_1(p,q)=\frac{1}{(p+q)^a},\quad a>-1,\ \mathrm{Re}(p,q)>0,\quad [2,\ \text{p. }136]$$

$$\phi_2(p,q)=\frac{\Gamma\left(a+\frac{1}{2}\right)\sqrt{(\pi pq)}}{(4p)^a}\ {}_2F_1\left(a,a+\frac{1}{2}\ ;2a+1\ ;\frac{p-q}{p}\right),\quad a>-1.\quad [2,\ \text{p. }160]$$

$$\therefore\ \frac{\Gamma\left(a+\frac{1}{2}\right)}{\sqrt{\pi}\,p^{2a}}\ {}_2F_1\left(a,a+\frac{1}{2}\ ;2a+1\ ;\frac{p^2-q^2}{p}\right)\doteqdot\begin{cases}\dfrac{x^{2a}}{\Gamma(a+1)}, & y>x\\[2mm] \dfrac{y^{2a}}{\Gamma(a+1)}, & y<x,\end{cases}$$

$a>-\frac{1}{2}$, $\mathrm{Re}(p,q)>0.$

4. Let

$$f(x,y)=\begin{cases}\dfrac{x^{a-1}}{\Gamma(a)}, & y>x\\[2mm] 0, & y<x,\end{cases}$$

$$\phi_1(p,q)=\frac{p}{(p+q)^a},\quad a>0,\ \mathrm{Re}(p,\ q)>0.\quad [2,\ \text{p. }136]$$

$$\phi_2(p,q)=\frac{\sqrt{\pi q}\quad\Gamma\left(a-\frac{1}{2}\right)\,2^{(1-2a)}}{p^{\left(a-\frac{3}{2}\right)}}\ {}_2F_1\left(a,a-\frac{1}{2}\ ;2a\ ;\frac{p-q}{p}\right),a>\frac{1}{2},\quad [2,\ \text{p. }160]$$

$$\therefore\ \frac{\Gamma\left(a-\frac{1}{2}\right)}{2\sqrt{\pi}\,p^{2(a-1)}}\,{}_2F_1\left(a,a-\frac{1}{2}\ ;2a\ ;\frac{p^2-q^2}{p^2}\right)$$

$$\doteqdot\begin{cases}\dfrac{x^{(2a-2)}}{\Gamma(a)}, & y>x\\[2mm] 0, & y<x,\end{cases}\qquad a>\frac{1}{2},\ \mathrm{Re}(p,q)>0.$$

5. Let

$$f(x,y)=\begin{cases}\dfrac{x^{(a-c)}\ y^b}{\Gamma(a-c)\Gamma(b-1)}\ {}_2F_1\left(c,c-a\ ;b+1\ ;\dfrac{y}{x}\right), & y>x\\[2mm] \dfrac{x^a y^{(b-c)}}{\Gamma(a+1)\Gamma(b-c)}\ {}_2F_1\left(c,c-b\ ;a+1\ ;\dfrac{x}{y}\right), & y<x.\end{cases}$$

$$\phi_1(p,q)=p^{(c-a)}\ q^{(c-b)}(p+q)^{-c},\quad a>-1,\ b>-1,\ \mathrm{Re}(p,q)>0.\quad [2,\ \text{p. }125]$$

$$\phi_2(p,q)=\frac{\Gamma\left(a+\frac{1}{2}\right)\,\Gamma\left(b+\frac{1}{2}\right)\,\Gamma(a+b-c+1)\sqrt{pq}}{\Gamma(a+b+1)\;q^{(b-c)}p^{a}}$$

$$\times{}_2F_1\left(c,a+\frac{1}{2}\;;\;a+b+1\;;\;\frac{p-q}{p}\right),$$

$$a>-\frac{1}{2},\quad b>-\frac{1}{2},\quad c<(a+b+1).\quad [2,\text{ p. }160]$$

$$\therefore\quad \frac{4^{(a+b-c)}\,\Gamma\left(a+\frac{1}{2}\right)\,\Gamma\left(b+\frac{1}{2}\right)\,\Gamma(a+b-c+1)}{\pi\Gamma(a+b+1)\;q^{2(b-c)}\;p^{2a}}$$

$$\times{}_2F_1\left(c,a+\frac{1}{2}\;;\;a+b+1\;;\;\frac{p^2-q^2}{p^2}\right)$$

$$\doteqdot\begin{cases}\dfrac{x^{2(a-c)}\;y^{2b}}{\Gamma(a-c)\Gamma(b-1)}\;{}_2F_1\left(c,c-a\;;\;b+1\;;\;\dfrac{y^2}{x^2}\right), & y>x\\[2ex] \dfrac{x^{2a}\,y^{2(b-c)}}{\Gamma(a+1)\Gamma(b-c)}\;{}_2F_1\left(c,c-b\;;\;a+1\;;\;\dfrac{x^2}{y^2}\right), & y<x,\end{cases}$$

$$a>-\frac{1}{2},\quad b>-\frac{1}{2},\quad c<a+b+1,\quad \mathrm{Re}(p,q)>0.$$

3. THEOREM 2. *Let*

$$f_1(x,y)\doteqdot\phi(p,q), \tag{3.1}$$

$$f_2(x,y)\doteqdot\sqrt{pq}\;f_1\left(\frac{1}{p},\;\frac{1}{q}\right), \tag{3.2}$$

then

$$xy\;f_2\left(\frac{x^2}{4},\;\frac{y^2}{4}\right)\doteqdot\frac{4}{\pi}\;\phi(p^2,\;q^2), \tag{3.3}$$

provided $f_2(x, y)$ *is absolutely integrable in* $0\leq x<\infty$, $0\leq y<\infty$, $\mathrm{Re}(p, q)>0$.

PROOF. From (3.1), we obtain

$$\phi(p,q)=pq\int_0^\infty\!\!\int_0^\infty e^{-px-qy}\,f_1(x,y)\;dx\;dy. \tag{3.4}$$

From (3.2), we have

$$\sqrt{pq}\;f_1\left(\frac{1}{p},\;\frac{1}{q}\right)=pq\int_0^\infty\!\!\int_0^\infty e^{-pu-qv}\;f_2(u,v)\;du\;dv.$$

$$\therefore\; f_1(x,y)=(xy)^{-\frac{1}{2}}\int_0^\infty\!\!\int_0^\infty e^{-\frac{u}{x}-\frac{v}{y}}\;f_2(u,v)\;du\;dv.$$

Making use of this equation, we obtain from (3.4)

$$\phi(p,q)=pq\int_0^\infty\int_0^\infty e^{-px-qy}(xy)^{-\frac{1}{2}}\,dx\,dy\int_0^\infty\int_0^\infty e^{-\frac{u}{x}-\frac{v}{y}}f_2(u,v)\,du\,dv.$$

Changing the order of integrations which is justified [1, p. 503] under the conditions mentioned in the theorem and evaluating the inner integral thus obtained, we get the required result on simplification.

APPLICATION.

Let
$$f_1(x,y)=\frac{2}{\sqrt{\pi}}\sqrt{x+y}.$$

$$\phi(p,q)=\frac{p^{\frac{3}{2}}-q^{\frac{3}{2}}}{\sqrt{pq}\,(p-q)},\quad [2,\ \text{p. }138]$$

$$f_2(x,y)=\begin{cases}\dfrac{2}{\pi\sqrt{x}}, & y>x\\[2mm] \dfrac{2}{\pi\sqrt{y}}, & y<x.\end{cases}\quad [2,\ \text{p. }136]$$

$$\therefore\ \frac{p^2+q^2+pq}{pq(p+q)}\doteqdot\begin{cases}y, & y>x\\ x, & y<x\end{cases},\quad \mathrm{Re}(p,q)>0.$$

Now we extend the results of theorems 1 and 2 respectively as:

4. THEOREM 3. *Let*

$$f(x,\ y)\doteqdot\phi_1(p,\ q),\tag{4.1}$$

$$\frac{1}{\sqrt{xy}}\ \phi_1\left(\frac{1}{x},\ \frac{1}{y}\right)\doteqdot\phi_2(p,\ q),\tag{4.2}$$

$$\frac{4}{\pi}\sqrt{xy}\ \phi_2\left(\frac{1}{4x^2},\ \frac{1}{4y^2}\right)\doteqdot\phi_3(p,\ q),\tag{4.3}$$

....

....

$$\frac{4}{\pi}\sqrt{xy}\ \phi_{n-1}\left(\frac{1}{4x^2},\ \frac{1}{4y^2}\right)\doteqdot\phi_n(p,\ q).\tag{4.4}$$

Then

$$f(x^{2^{n-1}},\ y^{2^{n-1}})\doteqdot\frac{4}{\pi pq}\phi_n\left(\frac{p^2}{4},\ \frac{q^2}{4}\right),\ (n=2,\ 3,\ \cdots,\ n)\tag{4.5}$$

provided $\frac{1}{\sqrt{xy}}f(x^{2^{n-2}},\ y^{2^{n-2}})$ *and* $f(x^{2^{n-2}},\ y^{2^{n-2}})$, $n=2,\ 3,\ \cdots,\ n$, *are absolutely integrable in* $0\leq x<\infty$, $0\leq y<\infty$, $\mathrm{Re}(p,q)>0$.

5. THEOREM 4. *Let*

$$f_1(x,y) \doteqdot \phi(p,q), \tag{5.1}$$

$$f_2(x,y) \doteqdot \sqrt{pq}\, f_1\left(\frac{1}{p}, \frac{1}{q}\right), \tag{5.2}$$

$$f_3(x,y) \doteqdot \frac{\pi}{4\sqrt{pq}} f_2\left(\frac{1}{4p^2}, \frac{1}{4q^2}\right), \tag{5.3}$$

....

....

$$f_n(x,y) \doteqdot \frac{\pi}{4\sqrt{pq}}\; f_{n-1}\left(\frac{1}{4p^2}, \frac{1}{4q^2}\right). \tag{5.4}$$

Then

$$xy\, f_n\left(\frac{x^2}{4}, \frac{y^2}{4}\right) \doteqdot \frac{4}{\pi}\, \phi(p^{2^{n-2}}, q^{2^{n-2}}), \quad (n=2, 3, \cdots, n) \tag{5.5}$$

provided $f_n(x, y)$, $n=2, 3, \cdots, n$, *are absolutely integrable in* $0 \leq x < \infty$, $0 \leq y < \infty$, $\mathrm{Re}(p, q) > 0$.

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