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ON A GENERALIZED FUNCTION OF n VARIABLES

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1. Introduction

The object of the present note is to define a generalized Special Function of n-variables. This function incorporates, as its special cases, Fox's H-function [2] and H-function of two variables introduced by Verma [8], which in turn includes the generalized function of two variables defined by Sharma [7] and Agarwal [1]. It also generalizes nearly all the known Special Functions of n variables, e. g., the Lauricella's functon F_A, F_B, F_C and F_D. Besides including the known Special Functions of n variables, as special cases, it leaves the possibility of defining through this new H-symbol, a great many Special Functions of n variables not so far recorded in the literature. It is expected that the study of this function will lead to very general, deeper and useful results in the theory of Special Functions. On account of the applications of the H-function of one variable in the various Statistical distributions, it is also expected that this function will play a very important role in the development of certain new properties of the Statistical distributions.

In what follows (x_n) denotes the sequence of n parameters $(x_1, x_2, \cdots, x_n)$ and the symbol (a_p, A_p) the sequence of p ordered pairs $(a_1, A_1), \cdots, (a_p, A_p)$.

2. Definition of $H^{A,\ (M_n : N_n)}_{B, C, (P_n : Q_n)}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

We define the generalized function of n-variables by means of n-fold multiple integral of Mellin-Barnes type as

$$H^{A, (M_n : N_n)}_{C, D, (P_n : Q_n)} \cdot \left[\begin{array}{c|c} x_1 & (c^*_j, \gamma_j) : (d_j, \delta_j) \\ x_2 & \\ \vdots & \\ x_n & (a_j^{(r)}, \alpha_j^{(r)}) : (b_j^{(r)}, \beta_j^{(r)}) \end{array}\right]$$

$$=H^{A,(M_n:N_n)}_{C,D,(P_n:Q_n)}\left[\begin{matrix}x_1\\x_2\\\vdots\\x_n\end{matrix}\left|\begin{matrix}(c^*_1,\gamma_1),\cdots,(c^*_C,\gamma_C),\ (d_1,\delta_1),\cdots,(d_D,\delta_D)\\ \\(a^{(1)}_{P_1},\alpha^{(1)}_{P_1}),\cdots,(a^{(n)}_{P_n},\alpha^{(n)}_{P_n}),\ (b^{(1)}_{Q_1},\beta^{(1)}_{Q_1}),\cdots,(b^{(n)}_{Q_n},\beta^{(n)}_{Q_n})\end{matrix}\right.\right]$$

$$=\frac{1}{(2\pi i)^n}\int_{-i\infty}^{+i\infty}\cdots\int_{-i\infty}^{+i\infty}\frac{\prod_{j=1}^{A}\Gamma(c^*_j-\sum_{r=1}^{n}s_r\gamma_j)}{\prod_{j=A+1}^{C}\Gamma(1-c^*_j+\sum_{r=1}^{n}s_r\gamma_j)\prod_{j=1}^{D}\Gamma(d_j-\sum_{r=1}^{n}s_r\delta_j)}$$

$$\times\prod_{r=1}^{n}\left\{\frac{\prod_{j=1}^{N_r}\Gamma(1-a_j^{(r)}+\alpha_j^{(r)}s_r)\prod_{j=1}^{M_r}\Gamma(b_j^{(r)}-s_r\beta_j^{(r)})(x_r)^{s_r}(ds_r)}{\prod_{j=1+N_r}^{Q_r}\Gamma(a_j^{(r)}-\alpha_j^{(r)}s_r)\prod_{j=1+M_r}^{N_r}\Gamma(1-b_j^{(r)}+s_r\beta_j^{(r)})}\right\}, \tag{2.1}$$

where $x_j\neq 0$ $(j=1,\cdots,n)$ and an empty product is interpreted as unity. Further A, C, $D: M_1,\cdots,M_n; N_1,\cdots,N_n; P_1,\cdots,P_n$ and $Q_1,\cdots,Q_n$ are integers satisfying the inequalities

$$0\leq A\leq C,\ 1\leq M_j\leq Q_j,\ 0\leq N_j\leq P_j\ (j=1,\cdots,n);$$

$a_j^{(r)}$'s, $b_j^{(r)}$'s, c_j^*'s and d_j' s are all complex numbers and α's, β's, γ' s and δ's are all positive numbers.

The sequence of parameters in the integrand of (2.1) are such that none of the poles coincide. That is the poles of the integrand of (2.1) are simple. In case the poles are not restricted to be simple, then by adopting a procedure due to Frobenius, the integral in (2.1) can be evaluated in terms of Psi-functions and generalized Zeta functions. In this connection, see [4] and [5] also.

The paths of integration are indented, if necessary, in such a manner that all the poles of $\Gamma(c^*_j-\sum_{r=1}^{n}s_r\gamma_j)$ for $j=1,\cdots,A$; and $\Gamma(b_j^{(r)}-s_r\beta_j^{(r)})$ for $j=1,\cdots,M_r$ and $r=1,\cdots,n$ are separated from the poles of $\Gamma(1-a_j^{(r)}+\alpha_j^{(r)}s_r)$ for $j=1,\cdots,N_r$ and $r=1,\cdots,n$.

From the asymptotic expansion of the gamma function, it readily follows that the integral in (2.1) converges if

$$\lambda_i<0,\ \mu_i>0,\ |\arg x_i|<(1/2)\pi\mu_i\ (i=1,\cdots,n)$$

where

$$\lambda_i=\sum_{j=1}^{C}\gamma_j+\sum_{j=1}^{P_i}\alpha_j^{(r)}-\sum_{j=1}^{D}\delta_j-\sum_{j=1}^{Q_i}\beta_j^{(r)}. \tag{2.2}$$

$$\mu_i=\sum_{j=1}^{M_i}\beta_j^{(r)}+\sum_{j=M_i+1}^{P_i}\beta_j^{(r)}+\sum_{j=1}^{N_i}\alpha_j^{(r)}-\sum_{j=N_i+1}^{P_i}\alpha_j^{(r)}+\sum_{j=1}^{A}\gamma_j-\sum_{j=A+1}^{C}\gamma_j-\sum_{j=1}^{D}\delta_j. \tag{2.3}$$

Whenever there is no risk of ambiguity, the generalized function of n-variables represented by the integral (2.1), will be denoted by the contrated notation

$$H^{A,(M_n:N_n)}_{C,D,(P_n:Q_n)}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

or simply by $$H\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

3. Asymptotic expansion of $H\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

The behaviour of the generalized function of n-variables in the vicinity of $x_1=x_2=\cdots=x_n=0$, is given by the relation

$$H\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0\left\{\prod_{j=1}^{n}|(x_j)|^{\phi_j}\right\}, \tag{3.1}$$

where

$$\phi_j=\min\{b_r^{(j)}/\beta_r^{(j)}\}, \qquad (r=1,2,\cdots,\ M_j). \tag{3.2}$$

On the other hand, when $|x_j|\to\infty$ ($j=1,\ \cdots,\ n$), the associated function $H^{(1)}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, which corresponds to the case $A=0$ of $H\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, has the behaviour

$$H^{(1)}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = 0\left\{\prod_{j=1}^{n}|x_j|^{\phi_j}\right\}, \tag{3.3}$$

where

$$\phi_j=\max\{(1-a_r^{(j)})/\alpha_r^{(j)}\} \qquad (r=1,\ \cdots,\ N_j).$$

4. Interesting particular cases

The following results are immediate consequences of the definition (2.1).

(i) When $n=2$, (2.1) reduces to Verma's H-function of two variables [8] which itself is a generalization of Agarwal's G-function of two variables [1].

(ii) On the other hand, for $A=C=D=0$, the generalized function of n-variables breaks up into a product of n H-functions.

We thus have

$$H^{0,(M_n:N_n)}_{0,0,(P_n:Q_n)}\left[\begin{matrix}x_1\\x_2\\\vdots\\x_n\end{matrix}\left|\begin{matrix}-;-\\\{a_j^{(r)},\alpha_j^{(r)}\}:\{b_j^{(r)},\beta_j^{(r)}\}\end{matrix}\right.\right]$$

$$=\prod_{j=1}^{n} H^{M_j,\ N_j}_{P_j,\ Q_j}\left[x_j\left|\begin{matrix}(a_{P_j}^{(j)},\ \alpha_{P_j}^{(j)})\\(b_{Q_j}^{(j)},\ \beta_{Q_j}^{(j)})\end{matrix}\right.\right] \tag{4.1}$$

5. A Single integral representation

The formula to be proved here is

$$\int_0^\infty \prod_{r=1}^{s} H^{M_r,\ N_r}_{P_r,\ Q_r}\left[x_r t\left|\begin{matrix}(a_{P_j}^{(r)},\ \alpha_{P_j}^{(r)})\\(b_{Q_j}^{(r)},\ \beta_{Q_j}^{(r)})\end{matrix}\right.\right]$$

$$\times H^{A,\ 0}_{C,\ D}\left[yt\left|\begin{matrix}(d_D,\ \delta_D)\\(c^*{}_C,\ \gamma_D)\end{matrix}\right.\right]dt$$

$$=H^{A,\ (M_s:N_s)}_{C,\ D:(P_s:Q_s)}\left[\begin{matrix}(x_{1/y})\\\vdots\\(x_{s/y})\end{matrix}\left|\begin{matrix}(c^*{}_j+\gamma_j,\ \gamma_j):(\delta_j+d_j,\delta_j)\\(a_j^{(r)},\ \alpha_j^{(r)}):(b_j^{(r)},\ \beta_j^{(r)})\end{matrix}\right.\right] \tag{5.1}$$

where $\mathrm{R}[\sum_{r=1}^{\infty}\{\min(b_1^{(r)})+\cdots+\min(b_{M_j}^{(r)})\}+\min(c_k/\gamma_k)+1]>0$, $(j=1,\ \cdots,\ s:k=1,\ \cdots,\ M)$; $\lambda_i<0$, $|\arg x_j|<(1/2)\pi\mu_j(j=1,\ \cdots,\ s)$ and $|\arg y|<(1/2)\pi\sigma$,

where
$$\sigma=\sum_{j=1}^{M}\gamma_j-\sum_{j=M+1}^{Q}\gamma_j-\sum_{j=1}^{N}\delta_j>0.$$

(5.1) readily follows on using the definition of the H-function of one variable [2, p. 239] in its integrand and applying the relation (2.1).

A special case of (5.1) for $s=2$ has already been given by the author [6, p. 187].

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REFERENCES

[1] R. P. Agarwal, *An extension of Meijer's G-function,* Proc. Nat. Inst. Sci. India Part A 31 (1965), 536—546.

[2] B. L. J. Braaksma, *Asymptotic expansions and analytic continuations for a class of Barnes integrals,* Compositio Math. 15 (1964) 239—341.

[3] A. Erdélyi, et al, *Higher transcendental functions,* Vol. I, McGraw-Hill, New York, 1953.

[4] A. M. Mathai, *Statistical theory of distribution and Meijer's G-function,* Metron 28 (1970), 122—146.

[5] A. M. Mathai and R. K. Saxena, *Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences,* Monograph, Dept. of Mathematics, McGill University, July, 1972.

[6] R. K. Saxena, *Integrals of products of H-functions.* Univ. Nac. Tucuman Rev. Ser. 21 (1971), 185—192.

[7] B. L. Sharma, *On the generalized function of two variables—I,* Ann. Soc. Sci. Bruxelles. Ser. I /79(1965), 26—40.

[8] R. U. Verma, *On the H-function of two variables—II.* An. Sti. Univ. 'Al—I—Cuza', Iasi, 17(1971), 103—109.