Kyungpook Math. J.
Volume 14, Number 2
December, 1974

INVARIANT SUBMANIFOLD OF A $\phi(4\pm2)$ STRUCTURE MANIFOLDS

By H. B. Pandey and R. S. Mishra

1. Introduction

Let V^m be a C^∞ m-dimensional Riemannian manifold imbedded in a C^∞ n-dimensional Riemannian manifold M^n, where $m<n$. The imbedding being denoted by

$$f: V^m \longrightarrow M^n.$$

Let B be the mapping induced by f i.e., $B=df$

$$df: T(V) \longrightarrow T(M).$$

Let $T(V,M)$ be the set of all vectors tangent to the submanifold $f(V)$, it is well known that

$$B: T(V) \longrightarrow T(V,M)$$

is an isomorphism. The set of all vectors normal to $f(V)$ forms a vector bundle over $f(V)$ which we shall denote by $N(V,M)$. We call $N(V,M)$ the normal bundle of V^m. The vector bundle induced by f from $N(V,M)$ is denoted by $N(V)$. We denote by $C: N(V) \longrightarrow N(V,M)$ the natural isomorphism and by $\eta^r_s(V)$ the space of all C^∞ tensor fields of type (r,s) associated with $N(V)$[1]. Thus $\mathscr{L}^0_0(V) = \eta^0_0(V)$ is the space of all C^∞ functions defined on V^m while an element of $\eta^1_0(V)$ is a C^∞ vector field normal to V^m and an element of $\mathscr{L}^1_0(V)$ is a C^∞ vector field tangential to V^m.

Take vector fields $\overline{X}$ and $\overline{Y}$ defined along $f(V)$. Let $\tilde{X}$ and $\tilde{Y}$ be the local extensions of $\overline{X}$ and $\overline{Y}$ respectively, then $[\tilde{X},\tilde{Y}]$ is a vector field tangential to M^n and its restriction $[\tilde{X},\tilde{Y}]/f(V)$ to $f(V)$ is determined independently of the choice of these local extensions $\tilde{X}$ and $\tilde{Y}$. Therefore we can define $[\overline{X},\overline{Y}]$ by

(1.1) $$[\overline{X},\overline{Y}] = [\tilde{X},\tilde{Y}]/f(V)$$

Since B is an isomorphism

(1.2) $$[BX,BY] = B[X,Y]$$

1) We denote by $\mathscr{L}^r_s(V)$ the space of all C^∞ tensor fields of type (r,s) associated with $T(V)$.

holds for all $X\in\mathscr{L}_0^1(V)$ and $Y\in\mathscr{L}_0^1(V)$.

Let $\tilde{G}$ be the Riemannian metric tensor of M^n. We define g and g^* on V^m and $N(V)$ respectively as follows

$$g(X_1, X_2)=\tilde{G}(BX_1, BX_2)f \tag{1.2}$$

and

$$g^*(N_1, N_2)=G(CN_1, CN_2)$$

for all $X_1, X_2\in\mathscr{L}_0^1(V)$ and $N_1, N_2\in\eta_0^1(V)$.

It can be verified that g is a Riemannian metric tensor in V^m which is called the induced metric tensor in V^m and g^* is a tensor field defining an inner product in $N(V)$. The tensor g^* is called the induced metric of $N(V)$.

Let $\tilde{\nabla}$ be the Riemannian connexion determined by $\tilde{G}$ in M^n, then $\tilde{\nabla}$ induces a connexion ∇ in $f(V)$ defined by

$$\nabla_{\bar{X}}\bar{Y}=(\tilde{\nabla}_{\tilde{X}}\tilde{Y})/f(V), \tag{1.4}$$

where $\bar{X}$ and $\bar{Y}$ are arbitrary C^∞ vector fields defined along $f(V)$ and tangential to $f(V)$. Thus taking account of (1.1) we have

$$\nabla_{\bar{X}}\bar{Y}-\nabla_{\bar{Y}}\bar{X}=[\bar{X}, \bar{Y}] \tag{1.5}$$

Let us suppose in the present paper that M^n is a C^∞ $\phi(4,2)$ structure manifold with structure tensor $\tilde{\phi}$ of type (1.1)*1. Let $\tilde{L}$ and $\tilde{M}$ be the complementary distributions corresponding to the projection operators $\tilde{l}$ and $\tilde{m}$ respectively, where

$$\tilde{l}=-\tilde{\phi}^2,\ m=I+\tilde{\phi}^2 \tag{1.6}$$

and I denotes the identity operator. These operators satisfy the following relations

$$\tilde{\phi}\tilde{l}=\tilde{l}\tilde{\phi}=-\tilde{\phi}^3,\ \tilde{\phi}\tilde{m}=\tilde{m}\tilde{\phi}=\tilde{\phi}^3+\tilde{\phi} \tag{1.7}$$
$$\tilde{\phi}^2\tilde{l}=-\tilde{l}^2=-\tilde{l},\ \tilde{\phi}^2\tilde{m}=\tilde{m}\tilde{\phi}^2=0.$$

Such a manifold M^n always admits a Riemannian metric say $\tilde{G}$ which satisfies the following relation

$$\tilde{G}(\tilde{\phi}^3\tilde{X}, \tilde{\phi}^3\tilde{Y})=\tilde{G}(\tilde{\phi}^2\tilde{X}, \tilde{\phi}^2\tilde{Y})$$

for any two vector fields $\tilde{X}, \tilde{Y}$.

2. Invariant submanifold in a $\phi(4,2)$ structure manifold

Let V^m be a C^∞ m-dimensional manifold imbedded as a submanifold in a C^∞ n-dimensional $\phi(4,2)$ structure manifold M^n with (1,1) structure tensor $\tilde{\phi}$. V^m is

*1 The (1,1) type tensor field $\tilde{\phi}$ satisfies $\tilde{\phi}^4+\tilde{\phi}^2=0$, [1].

defined to be an invariant submanifold of M^n, if the tangent space $T_p(f(v))$ of $f(V)$ is invariant by the linear mapping $\tilde{\phi}$ at each point p of $f(V)$.

Throughout this paper we assume V^m to be an invariant submanifold of M^n, so that for $X\in\mathscr{L}^1_0(V)$ we have

(2.1) $$\tilde{\phi}BX=B\,\phi X$$

where ϕ is a (1, 1) tensor field in V^m.

Let us denote by $\tilde{N}$ and N the Nijenhuis tensors in M^n and V^m determined by the (1, 1) tensor fields $\tilde{\phi}$ and ϕ respectively.

THEOREM 2.1. *The Nijenhuis tensor $\tilde{N}$ and N are related as*

(2.2) $$\tilde{N}(BX, BY)=BN(X, Y)$$

PROOF. We have

$$\begin{aligned}\tilde{N}(BX, BY)&=[\tilde{\phi}BX, \tilde{\phi}BY]-\tilde{\phi}[BX, \tilde{\phi}BY]-\tilde{\phi}[\tilde{\phi}BX, BY]+\tilde{\phi}^2[BX, BY]\\&=[B\phi X, B\phi Y]-\tilde{\phi}[BX, B\phi Y]-\tilde{\phi}[B\phi X, BY)+\tilde{\phi}^2[BX, BY]\\&=B[\phi X, \phi Y]-\tilde{\phi}B[X, \phi Y]-\tilde{\phi}B[\phi X, Y]+\tilde{\phi}^2B[X, Y]\\&=BN(X, Y).\end{aligned}$$

Particular cases. Let us consider the following two cases for any invariant submanifold V^m in a $\phi(4, 2)$ structure manifold M^n.

Case 1. The distribution $\tilde{M}$ is never tangential to $f(V)$ i.e., no vector field of the type $\tilde{m}\bar{X}$ where $\bar{X}$ is a vector field tangential to $f(V)$ is tangential to $f(V)$. Later it will be proved that in this case V^m is necessarily even dimensional.

Case 2. The distribution $\tilde{M}$ is always tangential to $f(V)$.

First of all we will consider case 1.

The distribution $\tilde{M}$ is never tangential to the invariant submanifold $f(V)$, implies any vector field of the type $\tilde{m}\bar{X}$ is independent of any vector field of the form $BX, X\in\mathscr{L}^1_0(V)$. Applying ϕ to (2.1) we get

(2.3) $$\tilde{\phi}^2BX=B\phi^2X$$

We now show that the vector fields of type $BX, X\in\mathscr{L}^1_0(V)$ are in the distribution $\tilde{L}$, which is equivalent to showing that $\tilde{m}(BX)=0$. Suppose

$$\tilde{m}(BX)\neq 0.$$

In view of (1.6) we have

$$\begin{aligned}\tilde{m}(BX)&=(I+\tilde{\phi}^2)BX\\&=BX+B\phi^2X\\&=B(X+\phi^2X)\end{aligned}$$

This relation shows that $\tilde{m}(BX)$ is tangential to $f(V)$ which contradicts the hypothesis hence

$$\tilde{m}(BX)=0$$

Hence, using (1.7) in (2.3) we get

$$B\phi^2X=-BX$$

which in view of B being an isomorphism yields

$$\phi^2X=-X \tag{2.4}$$

Consequently the (1,1) tensor field ϕ in V^m, is an almost complex structure, called induced almost complex structure on the invariant submanifold V^m.

Next, we define a tensor field $\tilde{H}$(called the Haantjes tensor) of type (1,2) in M^n as follows

$$\tilde{H}(\tilde{X},\tilde{Y})=\tilde{N}(\tilde{X},\tilde{Y})-\tilde{N}(\tilde{m}\tilde{X},\tilde{Y})-\tilde{N}(\tilde{X},\tilde{m}\tilde{Y})+\tilde{N}(\tilde{m}\tilde{X},\tilde{m}\tilde{Y}) \tag{2.5}$$

for any two vector fields $\tilde{X}$ and $\tilde{Y}\in\mathscr{L}^1_0(M)$.

THEOREM 2.2. *The* (1,2) *tensor field* H *defined in* M^n *satisfies*

$$\tilde{H}(BX,BY)=\tilde{N}(BX,BY)=BN(X,Y) \tag{2.6}$$

for all $X,Y\in\mathscr{L}^1_0(V)$.

PROOF. Since any vector field tangential to $f(V)$ is not contained in the distribution $\tilde{M}$, we have for any $X\in\mathscr{L}^1_0(V)$

$$\tilde{m}(BX)=0$$

which in view of (2.5) and (2.2) yields

$$\tilde{H}(BX,BY)=BN(X,Y).$$

Combining the above results we can state:

THEOREM 2.3. *An invariant submanifold* V^m *imbedded in a* $\phi(4,2)$ *structure manifold such that the distribution* $\tilde{M}$ *is never tangential to* $f(V)$ *is an almost complex manifold with induced almost complex structure* ϕ. *Consequently the dimension of* V^m *is even. If in the* $\phi(4,2)$ *structure manifold Haantjes tensor vanishes then the invariant submanifold is complex.*

Case 2. The distribution $\tilde{M}$ is always tangential to the invariant submanifold $f(V)$ implies for each $X\in\mathscr{L}^1_0(V)$

$$\tilde{m}(BX)=BmX. \tag{2.7}$$

Again we define a (1,1) tensor field in V^m by

$$l=-\phi^2. \tag{2.8}$$

Thus

$$lX = -\phi^2 X$$

for all $X \in \mathscr{L}_0^1(V)$. Applying B on both sides we get

$$BlX = -B\phi^2 X$$
$$= -\phi^2 BX$$

(2.9) $$BlX = \tilde{l}BX$$

THEOREM 2.4. (1, 1) *tensor fields l and m in V^m defined by* (2.7) *and* (2.9) *satisfy the following relations*

(2.10) $$l + m = I, \ lm = ml = 0, \ l^2 = l, \ m^2 = m.$$

PROOF. Since

$$\tilde{l} + \tilde{m} = I$$

Operating on a vector of the type BX, $X \in \mathscr{L}_0^1(V)$, we get

$$\tilde{l}BX + \tilde{m}BX = BX$$

which in view of (2.9) and (2.7) is equivalent to

$$BlX + BmX = BX$$

Since B is an isomorphism the above equation yields

$$lX + mX = X.$$

That is

$$l + m = I$$

Next since

$$\tilde{l}\tilde{m} = \tilde{m}\tilde{l} = 0$$

operating $\tilde{l}\tilde{m}$ and $\tilde{m}\tilde{l}$ on BX, $X \in \mathscr{L}_0^1(V)$ and using (2.7) and (2.9) we get

(2.11) $$BlmX = 0 \text{ and } BmlX = 0$$

which implies

$$lmX = 0 = mlX$$

or

$$lm = 0, \ ml = 0$$

Again we have

$$\tilde{l}^2 = \tilde{l}$$

and

$$\tilde{m}^2 = \tilde{m}$$

Operating $\tilde{l}^2, \tilde{m}^2$ on BX we get

$$\tilde{l}^2 BX = \tilde{l}BX$$
$$Bl^2 X = BlX$$

and

$$Bm^2X=BmX.$$

Hence
$$l^2X=lX$$

and

$$m^2X=mX$$

which yields

$$l^2=l \text{ and } m^2=m.$$

The relation (2.10) shows that l and m are complementary projection operators in V^mgiven by

$$l=-\phi^2,\quad m=I+\phi$$

we have by virtue of (2.1)

$$\begin{aligned}B\phi^4X&=\tilde{\phi}^4BX\\&=-\tilde{\phi}^2BX\\&=-B\phi^2X\end{aligned}$$

which yields

$$\phi^4+\phi^2=0.$$

Hence ϕ acts as an $\phi(4,2)$ structure on V^m called the induced $\phi(4,2)$ structure on V^m.

THEOREM 2.5. *We have*

(2.12) $$\tilde{H}(BX,BY)=BH(X,Y)$$

PROOF. In view of (2.2) we get

$$\begin{aligned}\tilde{H}(BX,BY)&=BN(X,Y)-BN(mX,Y)-BN(X,mY)+BN(mX,mY)\\&=BH(X,Y)\end{aligned}$$

Hence the result follows.

In the light of above results we can state:

THEOREM 2.6. *An invariant submanifold V^m imbedded in an $\phi(4,2)$ structure manifold M^n in such a way that the distribution $\tilde{M}$ is always tangential to $f(V)$ is an $\phi(4,2)$ structure manifold with induced structure ϕ. If the Haantjes tensor vanishes in M^n then it vanishes in V^m also.*

It is well known [5] that the necessary and sufficient condition for L to be integrable is

(2.13) $$mN(\phi X,\phi Y)+\phi mN(\phi X,Y)+\phi mN(X,\phi Y)=0.$$

Next we have

$$\tilde{m}\tilde{N}(\tilde{\phi}BX,\tilde{\phi}BY)+\tilde{\phi}\tilde{m}\tilde{N}(\tilde{\phi}BX,BY)+\tilde{\phi}\tilde{m}\tilde{N}(BX,\tilde{\phi}BY)$$
$$=B[mN(\phi X,\phi Y)+\phi mN(\phi X,Y)+\phi mN(X,\phi Y)]. \tag{2.14}$$

THEOREM 2.7. *If the distribution $\tilde{L}$ is integrable in M^n then the distribution L is integrable in V^m.*

PROOF. It follows from (2.13) and (2.14).

It is well known [5] that the necessary and sufficient condition for M to be integrable is

$$\phi^2N(X,Y)-\phi^2N(\phi X,\phi Y)-\phi^3N(\phi X,Y)-\phi^3N(X,\phi Y)=0. \tag{2.15}$$

THEOREM 2.8. *If the distribution $\tilde{M}$ is integrable in M^n then the distribution M is integrable in V^m.*

PROOF. We have

$$\tilde{\phi}^2\tilde{N}(BX,BY)-\tilde{\phi}^2\tilde{N}(\tilde{\phi}BX,\tilde{\phi}BY)-\tilde{\phi}^3\tilde{N}(\tilde{\phi}BX,BY)-\tilde{\phi}^3N(BX,\tilde{\phi}BY)$$
$$=B[\phi^2N(X,Y)-\phi^2N(\phi X,\phi Y)-\phi^3N(\phi X,Y)-\phi^3N(X,\phi Y). \tag{2.16}$$

From this the result follows.

3. Invariant submanifold of $\phi(4,-2)$ structure manifold

Let M^n be an n dimensional differentiable manifold of class C^∞ and let there be given a tensor field $\tilde{\phi}(\neq0)$ of type (1,1) and of class C^∞ such that

$$\tilde{\phi}^4-\tilde{\phi}^2=0. \tag{3.1}$$

Let $\tilde{l}'$ and $\tilde{m}'$ be the projection operators defined as

$$\tilde{l}'=\tilde{\phi}^2,\quad \tilde{m}'=I-\tilde{\phi}^2 \tag{3.2}$$

where I is the identity operator.

Let $\tilde{L}'$ and $\tilde{M}'$ be the complementary distributions corresponding to the projection operators given by (3.2). These operators satisfy the following relations:

$$\tilde{\phi}\tilde{l}'=\tilde{\phi}^3=\tilde{l}'\tilde{\phi},\quad \tilde{\phi}\tilde{m}'=\tilde{m}'\tilde{\phi}=\tilde{\phi}-\tilde{\phi}^3$$
$$\tilde{\phi}^2\tilde{l}'=\tilde{\phi}^2=\tilde{l}',\quad \tilde{\phi}^2\tilde{m}'=\tilde{m}'\tilde{\phi}^2=0. \tag{3.3}$$

Throughout this section let us assume M^n to be a $\phi(4,-2)$ structure manifold. Let V^m be a C^∞ m dimensional manifold imbedded as a submanifold in a C^∞ n-dimensional manifold M^n. V^m is defined to be an invariant submanifold of M^n if the tangent space $T_p(f(V))$ of $f(V)$ is invariant by the linear mapping $\tilde{\phi}$ at each point p of $f(V)$.

Let us assume V^m to be an invariant submanifold of M^n, so that $X\in\mathscr{L}^1_0(V)$ we have

$$\tilde{\phi}BX=B\phi X \tag{3.4}$$

where ϕ is (1, 1) tensor field in V^m.

Let us denote $\tilde{N}$ and N the Nijenhuis tensors in M^n and V^m determined by $\tilde{\phi}$ and ϕ respectively. It can be easily verified that

$$\tilde{N}(BX, BY) = BN(X, Y). \tag{3.5}$$

Particular cases. Let us consider the following two cases for any invariant submanifold V^m in a $\phi(4, -2)$ structure manifold M^n.

Case 1. The distribution $\tilde{M}'$ is never tangential to $f(V)$ i.e., no vector field of the type $\tilde{m}'\bar{X}$ where $\bar{X}$ is a vector field tangential to $f(V)$ is tangential to $f(V)$.

Case 2. The distribution $\tilde{M}'$ is always tangential to $f(V)$.

Let us take case 1. The distribution $\tilde{M}'$ is never tangential to the invariant submanifold $f(V)$ implies any vector field of the type $\tilde{m}'\bar{X}$ is independent of any vector field of the form BX, $X \in \mathscr{L}^1_0(V)$. Applying ϕ to (3.4) we get

$$\tilde{\phi}^2 BX = B\phi^2 X \tag{3.6}$$

We now show that the vector fields of type $BX, X \in \mathscr{L}^1_0(V)$ are in the distribution $\tilde{L}'$ which is equivalent to showing that

$$m'(BX) = 0.$$

Suppose

$$m'(BX) \neq 0.$$

In view of (3.2) we have

$$\begin{aligned} \tilde{m}'(BX) &= (I - \tilde{\phi}^2)(BX) \\ &= BX - B\phi^2 X \\ &= B(X - \phi^2 X) \end{aligned}$$

This relation shows that $\tilde{m}'(BX)$ is tangential to $f(V)$ which contradicts the hypothesis. Hence

$$\tilde{m}'(BX) = 0.$$

Using (3.3) in (3.6) we get

$$B\phi^2 X = BX$$

Since B is an isomorphism, hence

$$\phi^2 X = X \tag{3.7}$$

Consequently the (1, 1) tensor field ϕ in V^m, is an almost product structure, called induced almost product structure on the invariant submanifold V^m.

Let us define a (1, 2) type tensor field H (known as Haantjes tensor) in M^n as follows

(3.8) $$\tilde{H}(\tilde{X},\tilde{Y})\overset{\text{def}}{=\!=}\tilde{N}(\tilde{X},\tilde{Y})-\tilde{N}(\tilde{m}'\tilde{X},\tilde{Y})-N(\tilde{X},\tilde{m}',\tilde{Y})+\tilde{N}(\tilde{m}'\tilde{X},\tilde{m}'\tilde{Y})$$

for any two vector fields $X, Y\in\mathscr{L}^1_0(M)$.

It can be easily verified that

(3.9) $$\tilde{H}(BX, BY)=BN(X,Y).$$

In the light of the results obtained above, we can state:

THEOREM 3.1. *An invariant submanifold V^m imbedded in a $\phi(4, -2)$ structure manifold such that the distribution $\tilde{M}'$ is never tangential to $f(V)$ is equipped with almost product structure. If in the $\phi(4, -2)$ structure manifold Haantjes tensor vanishes then in the invariant submanifold Nijenhuis tensor vanishes.*

Case 2. The distribution $\tilde{M}'$ is always tangential to the invariant submanifold $f(V)$ implies for each $X\in\mathscr{L}^1_0(V)$.

(3.10) $$\tilde{m}'(BX)=Bm'X$$

Let us define a (1, 1) tensor field in V^m by

(3.11) $$l'=\phi^2$$

Thus

$$l'x=\phi^2X$$

for all $X\in\mathscr{L}^1_0(V)$. Applying B on both sides we get

(3.12) $$\begin{aligned}Bl'X&=B\phi^2X\\&=\tilde{\phi}^2BX\\&=\tilde{l}BX\end{aligned}$$

THEOREM 3.2. *The* (1, 1) *tensor fields l' and m' in V^m defined by* (3.10) *and* (3.12) *satisfy the following relations*

(3.13) $$l'+m'=I,\ l'm'=m'l'=0,\ l'^2=l',\ m'^2=m'.$$

PROOF. It follows the pattern of the proof of theorem (2.4).

The relation (3.13) shows that l' and m' are complementary projection operators in V^m given by

(3.14) $$l'=\phi^2,\ \ m'=I-\phi^2.$$

We have by virtue of (3.1)

$$\begin{aligned}B\phi^4X&=\tilde{\phi}^4BX\\&=\tilde{\phi}^2BX\\&=B\phi^2X\end{aligned}$$

which yields

(3.15) $$\phi^4-\phi^2=0.$$

Hence ϕ acts as an $\phi(4,-2)$ structure on V^m called the induced $\phi(4,-2)$ structure on V^m.

In this case we can easily verify that

(3.16) $$\tilde{H}(BX, BY)=BH(X,Y)$$

THEOREM 3.3. *An invariant submanifold V^m imbedded in an $\phi(4,-2)$ structure manifold M^n in such a way that the distribution $\tilde{M}$ is always tangential to $f(V)$ is an $\phi(4,-2)$ structure manifold, with induced structure ϕ. If the Haantjes tensor vanishes in M^n, then it vanishes in V^m also.*

It is well known [5] that the necessary and sufficient condition for L' to be integrable is

(3.17) $$m'N(\phi X,\phi Y)+\phi m'N(\phi X,Y)+\phi m'N(X,\phi Y)=0.$$

THEOREM 3.4. *If the distribution $\tilde{L}$ is integrable in M^n then the distribution L is integrable in V^m.*

PROOF. We have

(3.18) $$\tilde{m}'\tilde{N}(\tilde{\phi}BX,\tilde{\phi}BY)+\tilde{\phi}\tilde{m}'\tilde{N}(\tilde{\phi}BX,BY)+\tilde{\phi}\tilde{m}'\tilde{N}(BX,\tilde{\phi}BY)$$
$$=B[m'N(\phi X,\phi Y)+\phi m'N(\phi X,Y)+\phi m'N(X,\phi Y)$$

The result follows from (3.17) and (3.18).

It is also known that the necessary and sufficient condition for M' to be integrable is

(3.19) $$\phi^2N(X,Y)+\phi^2N(\phi X,\phi Y)+\phi^3N(\phi X,Y)+\phi^3N(X,\phi Y)=0$$

Hence we can state:

THEOREM 3.5. *If the distribution $\tilde{M}$ is integrable in M^n then the distribution M is integrable in V^m.*

PROOF. It follows the pattern of the proof of theorem (2.8).

Banaras Hindu University
Varanasi 221005, India

REFERENCES

[1] Yano, K.C.S. Houl and Chen, *Structures defined by a tensor* ϕ *of type* (1,1) *satisfying* $\phi^4 \pm \phi^2 = 0$, Tensor N.S. 23 (1972) 81—87.

[2] Yano, K., *Differential geometry on Complex and almost complex spaces*, Pergamon Press (1964).

[3] Clark R.S. and D.S. Geol, *Almost tangent structure manifolds, of the second order*, Tohoku Math. J. 24, 79—92.

[4] Yano, K. and S. Ishihara, *Invariant submanifolds of an almost contact manifold*, Kōdai Math. Sem. Rep. 21 (1969) 350—367.

[5) R.S. Mishra and H.B. Pandey, *Integrability of certain structures* (under print).