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ON WEAK CONTINUOUS FUNCTIONS INTO HAUSDORFF SPACES

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1. Introduction

It is a wellknown theorem that every continuous function from a compact space into any Hausdorff space is closed. This property is a characterizing property of functionally compact spaces (See [3], [5]). This theorem was treated by other auther in case of C-compact spaces [4].

In the present paper, we generalize this theorem by weakening continuity condition or compactness condition, and we have a characterization of absolutely closedness.

2. Generalizations of the theorem

DEFINITION 2.1. A function $f: X \to Y$ is *almost* (θ-, *weakly*) *continuous* ([6], [8]) if and only if for every $x \in X$ and every open neighborhood V of $f(x)$ there exists an open neighborhood U of x such that $f(U) \subset V^{-0}$ ($f(U^-) \subset V^-$, $f(U) \subset V^-$, respectively).

It is shown ([5], [6]) that

continuity $\Rightarrow$ almost continuity $\Rightarrow$ θ-continuity $\Rightarrow$ weak continuity,

but none of the above implications can be reversed.

DEFINITION 2.2. A topological space is called *C-compact* [4] if and only if given a closed subset Q of X and an open cover $\mathcal{O}$ of Q, then there exists a finite number of members of $\mathcal{O}$ whose closures cover Q.

DEFINITION 2.3. A topological space X is called *functionally compact* if and only if whenever $\mathscr{U}$ is an open filter base on X such that $\bigcap \mathscr{U} = \bigcap \{U^- | U \in \mathscr{U}\} (=A)$, then $\mathscr{U}$ is a neighborhood base of A.

DEFINITION 2.4. A topological space X is called *generalized absolutely closed* if and only if every open cover of X has a finite subfamily whose union is dense in X [7].

It is shown ([4], [3], [5]) that

compactness$\Rightarrow$$C$-compactness$\Rightarrow$functional compactness$\Rightarrow$generalized absolutely closedness, but none of these implications can be reversed.

We now have the following generalized theorems.

THEOREM 2.5. *Every weakly continuous function from a compact space into any Hausdorff space is closed.*

PROOF. Let f be a weakly continuous function from a compact space X into a Hausdorff space Y. Let F be a closed subset of X and let $y \notin f(F)$. For each $f(x)$ of $f(F)$, by Hausdorffness of Y, there exists an open neighborhood $V_{f(x)}$ of $f(x)$ such that $y \notin V^{-}_{f(x)}$. Again by weak continuity of f, there exists an open neighborhood U_x of x such that $f(U_x) \subset V^{-}_{f(x)}$. Since F is a closed subset of a compact space X, F is compact. Hence $F \subset \bigcup_{i=1}^{n} U_{x_i}$ for some finite x_i' s, and

$$f(F) \subset (\bigcup_{i=1}^{n} U_{x_i}) = \bigcup_{i=1}^{n} f(U_{x_i}) \subset \bigcup_{i=1}^{n} V^{-}_{f(x_i)}.$$

Thus $Y \sim \bigcup_{i=1}^{n} V^{-}_{f(x_i)}$ is open neighborhood of y which is disjoint from $f(F)$. Hence $f(F)$ is closed.

THEOREM 2.6. *Every θ-continuous function from a C-compact space into any Hausdorff space is closed.*

PROOF. Suppose that $f: X \to Y$ satisfies the hypothesis. Let Q be a closed subset of X and let $y \notin f(Q)$. Since Y is a Hausdorff space, each $f(x) \in f(Q)$ has an open neighborhood $V_{f(x)}$ such that $y \notin V^{-}_{f(x)}$. Since f is θ-continuous, there exists an open neighborhood U_x of x such that $f(U^{-}_x) \subset V^{-}_{f(x)}$. Now, since X is C-compact, $Q \subset \bigcup_{i=1}^{n} U^{-}_{x_i}$ for some finite x_i's. Hence

$$f(Q) \subset f(\bigcup_{i=1}^{n} U^{-}_{x_i}) = \bigcup_{i=1}^{n} f(U^{-}_{x_i}) \subset \bigcup_{i=1}^{n} V^{-}_{f(x_i)}.$$

Thus $Y \sim \bigcup_{i=1}^{n} V^{-}_{f(x_i)}$ is an open neighborhood of y which is disjoint from $f(Q)$. Hence $f(Q)$ is closed.

THEOREM 2.7. *Every almost continuous function from a functionally compact space into any Hausdorff space is closed.*

PROOF. In the propf of (i)⇒(ii) in Theorem 7 in [5], any other hypothesis on the space X is used except that X is functionally compact.

3. Generalized absolutely closed spaces

We have a similar theorem on generalized absolutely closed spaces and θ-continuity of a function into any Hausdorff space.

DEFINITION 3.1. A function $f: X \to Y$ is called *almost closed* if and only if for every regularly closed subset F (i.e. $F^{0-}=F$) of X, $f(F)$ is closed in Y.

THEOREM 3.2. *Every θ-continuous function from a generalized absolutely closed space into any Hausdorff space is almost closed.*

PROOF. Suppose that $f: X \to Y$ satisfies the hypothesis. Let C be a regularly closed subset of X and let $y \notin f(F)$. For each $f(c) \in f(C)$, by Hausdorffness of Y, there exists an open neighborhood $V_{f(c)}$ of $f(c)$ such that $y \notin V_{f(c)}^-$. Again by θ-contintinuity of f there exists an open neighborhood U_c of c such that $f(U_c^-) \subset V_{f(c)}^-$. Since a regularly closed subspace of a generalized absolutely closed space is again generalized absolutely closed ([2]) there exists a finite number of c_i ($i=1, 2, \cdots, n$) such that $C \subset \bigcup_{i=1}^{n} (U_i \cap C)^{-C} \cap \bigcup_{i=1}^{n} U_{c_i}^-$, where $-C$ denotes the closure operator in the subspace C. Hence

$$f(C) \subset \bigcup_{i=1}^{n} f(U_{c_i}^-) \subset \bigcup_{i=1}^{n} V_{f(c_i)}^-.$$

Thus $Y \sim \bigcup_{i=1}^{n} V_{f(c_i)}^-$ is an open neighborhood of y which is disjoint from $f(C)$. This proves that $f(C)$ is closed in Y.

DEFINITION 3.3. A Hausdorff space is called *absolutely closed* if and only if it cannot be properly imbedded in another Hausdorff space, or equivalently every open cover of the space has a finite subfamily whose union is dense in the space [7]

COROLLARY 3.4. *For a Hausdorff space X the following are equivalent.*

(1) *X is absolutely closed.*

(2) *Every θ-continuous function on X into any Hausdorff space is almost closed.*

(3) *Every almost continuous function from X into any Hausdorff space is almost closed.*

(4) *Every continuous function from X into any Hausdorff space is almost closed.*

PROOF. (1)$\Rightarrow$(2) : Theorem 3.2.

(2)$\Rightarrow$(3) : This is obvious from the fact that almost continuity implies θ-continuity

([5] Lemma 6).
(3)$\Rightarrow$(4) : This follows from the fact that continuity implies almost continuity [6].
(4)$\Rightarrow$(1) : If X is not absolutely closed, then X has an absolute closure κX [7]. Since the imbedding mapping $i : X \subset \kappa X$ is continuous and X is a regularly closed subset of X, by hypothesis $X=i(X)$ is closed in κX. This contradicts to the fact that $i(X)$ is dense in κX. Hence X is absolutely closed.

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