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MATRIX TRANSFORMATION IN THE CESARO SEQUENCE SPACES

By Kim-Pin Lim

Let $A=(a_{n,k})$ be an infinite matrix of complex numbers $a_{n,k}$ $(n,k=1,2,\cdots)$ and X, Y two subsets of the spaces of complex sequences. We say that the matrix A defines a matrix transformation from X into Y, if for every sequence $x=(x_k)\in X$ the sequence $A(x)=(A_n(x))$ is in Y, where $A_n(x)=\sum_{k=1}^{\infty} a_{n,k}x_k$. We denote the class of all matrix transformations from X into Y by (X, Y).

In [1], Leibowitz has shown that the Cesàro sequence spaces ces_p $(1<p<\infty)$ and ces_∞ with the norms

$$\|x\|'_p=\left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{\frac{1}{p}} \text{ for } 1<p<\infty$$

and

$$\|x\|'_\infty=\sup_n \frac{1}{n}\sum_{k=1}^{n}|x_k| \quad \text{for } p=\infty,$$

are Banach spaces, where $\text{ces}_p=\left\{x=(x_k) : \sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p<\infty\right\}$ and $\text{ces}_\infty=\left\{x=(x_k) : \sup_n \frac{1}{n}\sum_{k=1}^{n}|x_k|<\infty\right\}$. In this note, by taking a new slightly different norm, we determine the matrices of the classes (ces_p, l_∞) and (ces_p, c) where l_∞, c are respectively the spaces of bounded and convergent complex sequences. Meanwhile, we also determine all continuous linear functionals on ces_p for all $1<p<\infty$ and give a precise expression for $\|f\|$ with respect to this new norm, where f is a continuous linear functional on ces_p.

Now in the linear space ces_p, we define

$$\|x\|_p=\left(\sum_{r=0}^{\infty}\frac{1}{2^r}\sum_r |x_k|\right)^p\Bigg)^{\frac{1}{p}} \text{ for } 1<p<\infty$$

and

$$\|x\|_\infty=\sup_{r\geq 0}\frac{1}{2^r}\sum_r |x_k| \text{ for } p=\infty$$

where $\sum_r$ denotes a sum over the range $2^r\leq k<2^{r+1}$. Note that in case $p=\infty$,

ces_∞ is only a particular case in [2]. Therefore, we consider only the case $1<p<\infty$.

It is easy to check that $\| \|_p$ is a norm on ces_p. Furthermore, ces_p endowed with this norm is, in fact, a Banach space. For, if $(x^{(i)})$ is a Cauchy sequence in ces_p, then, obviously, $(x^{(i)})$ is a Cauchy sequence in ces_∞. Since ces_∞ is complete (see [3]), $x^{(i)}\to x\in\text{ces}_\infty$ as $i\to\infty$(i.e., $\sup_{r\geq 0}\frac{1}{2^r}\sum_r|x_k^{(i)}-x_k|\to 0$ as $i\to\infty$).

Therefore, $\varepsilon>0$, there exists a positive integer n_0 such that $\|x^{(n)}-x^{(m)}\|_p<\varepsilon^{\frac{1}{p}}$ $\forall n,m\geq N$. Therefore, for every t,

$$\sum_{r=0}^{t}\Big(\frac{1}{2^r}\sum_r|x_k^{(n)}-x_k^{(m)}|\Big)^p\leq\|x^{(n)}-x^{(m)}\|_p^p<\varepsilon.$$

Keeping n, and t fixed, let $m\to\infty$ then, we have

$$\sum_{r=0}^{t}\Big(\frac{1}{2^r}\sum_r|x_k^{(n)}-x_k|\Big)^p\leq\varepsilon,\ \text{for}\ n\geq N. \tag{1}$$

Since this is true for any t, it follows that

$$\sum_{r=0}^{\infty}\Big(\frac{1}{2^r}\sum_r|x_k^{(n)}-x_k|\Big)^p\leq\varepsilon\ \text{for}\ n\geq N. \tag{2}$$

Moreover,

$$\Big(\sum_{r=0}^{\infty}\Big(\frac{1}{2^r}\sum_r|x_k|\Big)^p\Big)^{\frac{1}{p}}\leq\Big(\sum_{r=0}^{\infty}\Big(\frac{1}{2^r}\sum_r|x_k^{(i)}|+\frac{1}{2^r}\sum_r|x_k^{(i)}-x_k|\Big)^p\Big)^{\frac{1}{p}}$$

$$\leq\Big(\sum_{r=0}^{\infty}\Big(\frac{1}{2^r}\sum_r|x_k^{(i)}|\Big)^p\Big)^{\frac{1}{p}}+\Big(\sum_{r=0}^{\infty}\Big(\frac{1}{2^r}\sum_r|x_k^{(i)}-x_k|\Big)^p\Big)^{\frac{1}{p}}$$

$$\leq\|x^{(i)}\|_p+\varepsilon^{\frac{1}{p}}\ \text{for}\ n\geq N \tag{3}$$

From (2) and (3) show that $x^{(i)}\to x\in\text{ces}_p(i\to\infty)$. Hence, cep_p is a Banach space. Moreover, it is easy to check that, for any $1<p<\infty$, $\|x\|_p\leq 2\|x\|'_p$. By bounded inverse theorem (see [4, p. 119]), $\| \|_p$ and $\| \|'_p$ are equivalent.

Finally, with regard to notation, the dual space of ces_p, i.e. the space of all continuous linear functionals on ces_p, will be denoted by ces^*_p and we write,

$$A_r(n)=\max_r|a_{n,k}|$$

where for each n the maximum is taken for k in $[2^r,\ 2^{r+1})$.

THEOREM 1. *Let* $1<p<\infty$. *Then* $A\in(\mathrm{ces}_p, l_\infty)$ *if and only if* $N(A)=$

$=\sup_n \left(\sum_{r=0}^{\infty}(2^r A_r(n))^q\right)^{\frac{1}{q}}<\infty$ *where* $q=\frac{p}{p-1}$.

PROOF. Suppose $N(A)<\infty$. For each n, we have

$$\begin{aligned}\sum_{k=1}^{\infty}|a_{n,k}x_k| &= \sum_{r=0}^{\infty}\sum_r |a_{n,k}x_k| \\ &\leq \sum_{r=0}^{\infty} 2^r A_r(n)\left(\frac{1}{2^r}\sum_r |x_k|\right) \\ &\leq \left(\sum_{r=0}^{\infty}(2^r A_r(n))^q\right)^{\frac{1}{q}}\left(\sum_{r=0}^{\infty}\left(\frac{1}{2^r}\sum_r |x_k|\right)^p\right)^{\frac{1}{p}} \\ &\leq N(A)\|x\|_p<\infty.\end{aligned}$$

This shows that $A\in(\mathrm{ces}_p, l_\infty)$.

Conversely, suppose that $A\in(\mathrm{ces}_p, l_\infty)$, then we have $\lim \sup_n|A_n(x)|<\infty$ for each $x\in\mathrm{ces}_p$. Thus, for each n and each $r\geq 0$, the functional

$$f_{r,n}(x)=\sum_r a_{n,k}x_k$$

are in ces_p^*, since

$$|f_{r,n}(x)|\leq A_r(n)\sum_r |x_k|\leq 2^r A_r(n)\|x\|_p.$$

It follows from the Banach-Steinhaus theorem for Banach space (see, [4, p. 115]) that, for each n, $\lim_{t\to\infty}\sum_{r=0}^{t} f_{r,n}(x)=A_n(x)$ is in ces_p^*, whence

$$|A_n(x)|\leq\|A_n\|\|x\|_p. \tag{4}$$

For each n, we take any integer $t>0$ and define $x\in\mathrm{ces}_p$ by $x_k=0$ for $k\geq 2^{t+1}$, and $x_{s(r)}=2^{rq}|a_{n,s(r)}|^{q-1}\mathrm{sgn}\, a_{n,s(r)}$, $x_k=0(k\neq s(r))$ for $0\leq r\leq t$ where $s(r)$ is such that $|a_{n,s(r)}|=\max_r |a_{n,k}|$. By (4), we get

$$\left|\sum_{r=0}^{t}2^{rq}A_r^q(n)\right|\leq\|A_n\|\left(\sum_{r=0}^{t}2^{rq}A_r^q(n)\right)^{\frac{1}{p}}$$

so that

$$\left(\sum_{r=0}^{t}2^{rq}A_r^q(n)\right)^{\frac{1}{q}}\leq\|A_n\|.$$

Since this inequality is true for any integer $t>0$, we have, for each n,

$$\left(\sum_{r=0}^{\infty}2^{rq}A_r^q(n)\right)^{\frac{1}{q}}\leq\|A_n\|<\infty. \tag{5}$$

Moreover, we have

$$|A_n(x)| \leq \Big(\sum_{r=0}^{\infty} 2^{rq} A_r^q(n)\Big)^{\frac{1}{q}} \|x\|_p \text{ for each } n.$$

This implies that

$$\|A_n\| \leq \Big(\sum_{r=0}^{\infty} 2^{rq} A_r^q(n)\Big)^{\frac{1}{q}}. \tag{6}$$

Together, (5) and (6) imply

$$\|A_n\| = \Big(\sum_{r=0}^{\infty} 2^{rq} A_r^q(n)\Big)^{\frac{1}{q}}.$$

Since $\limsup_n |A_n(x)| < \infty$ for each $x \in \text{ces}_p$, applying the Banach-Steinhaus theorem again, we have

$$\sup_n \|A_n\| = \sup_n \Big(\sum_{r=0}^{\infty} 2^{rq} A_r^q(n)\Big)^{\frac{1}{q}} < \infty$$

which proves the theorem.

We now characterize the class (ces_p, c).

THEOREM 2. *Let* $1 < p < \infty$. *Then* $A \in (\text{ces}_p, c)$ *if and only if*

(i) $a_{n,k} \to \alpha_k (n \to \infty,\ k \text{ fixed})$

(ii) $N(A) = \sup_n \Big(\sum_{r=0}^{\infty} 2^{rq} A_r^q(n)\Big)^{\frac{1}{q}} < \infty.$

PROOF. Suppose $A \in (\text{ces}_p, c)$. Then $A_n(x)$ exists for each $n \geq 1$ and $\lim_{n \to \infty} A_n(x)$ exists, for every $x \in \text{ces}_p$. Therefore, a similar argument to that in Theorem 1, we have the condition (ii). The condition (i) is obtained by taking $x = e_k \in \text{ces}_p$, where e_k is a sequence with 1 in the k^{th} place and zeros elsewhere.

On the other hand, suppose that the conditions (i) and (ii) are satisfied. Then a similar argument to that in Theorem 1, we get, for each $n \geq 1$,

$$|A_n(x)| < \infty \text{ for each } x \in \text{ces}_p.$$

Furthermore, the conditions (i) and (ii) imply

$$\Big(\sum_{r=0}^{\infty} 2^{rq} (\max_r |\alpha_k|)^q\Big)^{\frac{1}{q}} \leq N(A) < \infty,$$

where the maximum is taken for k in $[2^r, 2^{r+1})$. By using the fact that $\sum_{r=0}^{\infty}(2^r\max_r|\alpha_k|)^q$ is bounded, it is clear that $\sum_{k=1}^{\infty}\alpha_k x_k$ is absolutely convergent, for each $x\in\mathrm{ces}_p$. Moreover, for each $x\in\mathrm{ces}_p$, $\varepsilon>0$, we can choose an integer $m_0\geq 1$ such that

$$\sum_{r=m_0}^{\infty}\Big(\frac{1}{2^r}\sum_r|x_k|\Big)^p<\varepsilon^p.$$

Then, we have

$$\sum_{k=2^{m_0}}^{\infty}|a_{n,k}-\alpha_k||x_k|=\sum_{r=m_0}^{\infty}\sum_r|a_{n,k}-\alpha_k||x_k|$$
$$\leq\sum_{r=m_0}^{\infty}2^rB_r(n)\frac{1}{2^r}\sum_r|x_k|$$
$$\leq\Big(\sum_{r=m_0}^{\infty}2^{rq}B_r^q(n)\Big)^{\frac{1}{q}}\Big(\sum_{r=m_0}^{\infty}\Big(\frac{1}{2^r}\sum_r|x_k|\Big)^p\Big)^{\frac{1}{p}}$$
$$\leq M\varepsilon$$

where $\quad B_r(n)=\max_r|a_{n,k}-\alpha_k|$

and $\quad M=\sup_n\Big(\sum_{r=m_0}^{\infty}2^{rq}B_r^q(n)\Big)^{\frac{1}{q}}\leq 2N(A)<\infty.$

It follows immediately that $\lim_{n\to\infty}\sum_{k=1}^{\infty}a_{n,k}x_k=\sum_{k=1}^{\infty}\alpha_k x_k$, for each $x\in\mathrm{ces}_p$. This shows that $A\in(\mathrm{ces}_p, c)$ which proves the theorem.

Using the methods of Theorems 1 and 2, we can easily determine the dual space ces_p^* of $\mathrm{ces}_p(1<p<\infty)$, and also the norm of any element of ces_p^*.

THEOREM 3. *Let* $1<p<\infty$. *Then*

(i) $f\in\mathrm{ces}_p^*$ *if and only if there is a sequence* $(a_k)\in\mu$ *such that*

$$f(x)=\sum_{k=1}^{\infty}a_kx_k \tag{7}$$

and

$$\|f\|=\Big(\sum_{r=0}^{\infty}2^{rq}(\max_r|a_k|)^q\Big)^{\frac{1}{q}}$$

where $\mu=\Big\{(a_k):\sum_{r=0}^{\infty}2^{rq}(\max_r|a_k|)^q<\infty\Big\}$ *and* $\frac{1}{p}+\frac{1}{q}=1$.

(ii) ces_p^* *is equivalent to* μ.

PROOF. (i) As was shown in [1], for each $x \in \text{ces}_p$, we have $x=\sum_{k=1}^{\infty} x_k e_k$. Then, for every f in ces_p^*, we have

$$f(x)=\sum_{k=1}^{\infty} x_k f(e_k). \tag{8}$$

Put $a_k=f(e_k)$. A similar argument to that in Theorem 1, we get

$$\left(\sum_{r=0}^{\infty} 2^{rq} (\max_r |a_k|)^q\right)^{\frac{1}{q}} \leq \|f\| < \infty$$

i.e., $(a_k) \in \mu$. Furthermore, by (8),

$$\|f\| \leq \left(\sum_{r=0}^{\infty} 2^{rq} (\max_r |a_k|)^q\right)^{\frac{1}{q}}.$$

so that

$$\|f\| = \left(\sum_{r=0}^{\infty} 2^{rq} (\max_r |a_k|)^q\right)^{\frac{1}{q}}. \tag{9}$$

Thus, every f in ces_p^* can be written

$$f(x)=\sum_{k=1}^{\infty} x_k a_k$$

with $\|f\|$ given by (9), where $(a_k) \in \mu$.

Conversely, for any given $(a_k) \in \mu$ and the representation (7), obviously, defines an element of ces_p^*, which proves the part (i).

Furthermore, it is easy to see that the representation (7) is unique. Hence, we can define a mapping $T: \text{ces}_p^* \to \mu$ by

$$T(f)=(a_1, a_2, \cdots)$$

where a_k appear from the representation in (7). By part (i), we have

$$\|T(f)\| = \left(\sum_{r=0}^{\infty} 2^{rq} (\max_r |a_k|)^q\right)^{\frac{1}{q}} = \|f\|.$$

Thus T is norm preserving. It is surjective by the 'sufficiency' part of (i). Finally, T is obviously linear. Thus, ces_p^* can be identified with, i.e., is isometrically isomorphic to, the space μ, which proves the part (ii).

University of Malaya,
Kuala Lumpur,
Malaysia.

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