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AN EQUIVALENCE RELATION ON THE LATTICE OF TOPOLOGIES ON A SET

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1. Introduction

Kelley [1, Chapter 2] notes that two topologies on a set X are equal iff convergent nets and their limit points correspond in both topologies.

In this paper we consider the weaker condition that convergent nets but not necessarily limit points correspond in both topologies. This leads to an equivalence relation for topologies on X different from equality.

In this section we develop an equivalent condition in terms of topological structures which enables us to study the invariance of several topological properties.

Equivalence of topologies is formalized in the following

DEFINITION 1.1. Topologies $\mathscr{T}$ and $\mathscr{T}^*$ on X are *equivalent* (written $\mathscr{T}\equiv\mathscr{T}^*$) iff every $\mathscr{T}$-convergent net in X is $\mathscr{T}^*$-convergent and conversely.

That equivalent topologies need not be equal is shown in

EXAMPLE 1.2. Let $X=\{a,b\}$, $\mathscr{T}_1=\{\phi,X\}$, $\mathscr{T}_2=\{\phi,\{a\},X\}$, $\mathscr{T}_3=\{\phi,\{b\},X\}$ and $\mathscr{T}_4=\{\phi,\{a\},\{b\},X\}$. It is easy to see that $\mathscr{T}_1\equiv\mathscr{T}_2\equiv\mathscr{T}_3\not\equiv\mathscr{T}_4$.

THEOREM 1.3. *Let $\mathscr{T}$ and $\mathscr{T}^*$ be topologies on X, $\mathscr{T}$ being indiscrete. Then $\mathscr{T}^*\equiv\mathscr{T}$ iff there exists a point x in X such that X is the only $\mathscr{T}^*$-open set containing x.*

The proof of this theorem and several others in this paper follows from the following lemma whose proof is omitted.

LEMMA 1.4. *Let $D\neq\phi$ and $\geqq$ the relation on D defined by $d_1\geqq d_2$ for all d_1, d_2 in D. Then $(D,\geqq)$ is a directed set. We shall call $\geqq$ the trivial order on D.*

We continue now with the proof of Theorem 1.3.

Sufficiency. Suppose X is the only $\mathscr{T}^*$-open set containing x. Then every net in X is $\mathscr{T}^*$-convergent (to x). But every net in X is $\mathscr{T}$-convergent (to every point of X) and hence $\mathscr{T}^*\equiv\mathscr{T}$.

Necessity. Let $(D, \geqq)$ be the directed set in Lemma 1.4, with $D=X$ and let $i : D \to X$ be the identity net. Since the net $i : D \to X$ is $\mathcal{T}$-convergent and $\mathcal{T}^* \equiv \mathcal{T}$, it follows that $i : D \to X$ is $\mathcal{T}^*$-convergent to some point x in X. Let $x \in O^* \in \mathcal{T}^*$; then the net $i : D \to X$ is eventually in O^*. But $X = i[D] \subset O^*$ since $\geqq$ is the trivial order on D. Thus X is the only $\mathcal{T}^*$-open set containing x.

A similar argument proves the next result which uncovers the topological nature of our equivalence relation. This formulation is useful in the study of the invariance of many topological properties.

THEOREM 1.5. *Let $\mathcal{T}$ and $\mathcal{T}^*$ be topologies on X with closure operators* c *and* c* *and interior operators* Int *and* Int* *respectively. The following are equivalent*: (1) $\mathcal{T}^* \equiv \mathcal{T}$ (2) $\bigcap \{cA_\alpha : \alpha \in \Delta\} = \phi$ *iff* $\bigcap \{c^*A_\alpha : \alpha \in \Delta\} = \phi$ *for all collections* $\{A_\alpha : \alpha \in \Delta\}$ (3) $X = \bigcup \{\mathrm{Int} A_\alpha : \alpha \in \Delta\}$ *iff* $X = \bigcup \{\mathrm{Int}^* A_\alpha : \alpha \in \Delta\}$ *for all collections* $\{A_\alpha : \alpha \in \Delta\}$.

PROOF. We show only the equivalence of (1) and (2), the equivalence of (2) and (3) being obvious.

Suppose (1) holds and let $\{A_\alpha : \alpha \in \Delta\}$ be a family of sets for which $\bigcap \{cA_\alpha : \alpha \in \Delta\} \neq \phi$. We will show that $\bigcap \{c^*A_\alpha : \alpha \in \Delta\} \neq \phi$. Let $x \in \bigcap \{cA_\alpha : \alpha \in \Delta\}$; then $N \cap A_\alpha \neq \phi$ for all $\alpha \in \Delta$ and all $N \in \eta(x, \mathcal{T})$. Let $(D, \geqq)$ be the directed set in Lemma 1.4 with $D = \Delta$ and let $\geqq'$ be the usual ordering on $\eta(x, \mathcal{T})(N_1 \geqq' N_2$ iff $N_1 \subset N_2)$. Let $D^* = D \times \eta(x, \mathcal{T})$ and let $\geqq^*$ be the product ordering. Let $S : D^* \to X$ as follows: $S(\alpha, N) \in N \cap A_\alpha$. It is clear that S is $\mathcal{T}$-convergent to x. Since $\mathcal{T}^* \equiv \mathcal{T}$, there exists a point y in X such that S is $\mathcal{T}^*$-convergent to y. We complete the proof by showing that $y \in c^*A_\alpha$ for all $\alpha \in \Delta$ or that $N^* \cap A_\alpha \neq \phi$ for all $\alpha \in \Delta$ and all $N^* \in \eta(y, \mathcal{T}^*)$. Fix $\alpha^* \in \Delta$ and $N^* \in \eta(y, \mathcal{T}^*)$. Since S is eventually in N^*, there exists a pair $(\alpha, N) \in D^*$ such that $S(\beta, M) \in N^*$ when $(\beta, M) \geqq^* (\alpha, N)$. In particular, $(\alpha^*, N) \geqq^* (\alpha, N)$ and hence $S(\alpha^*, N) \in N^*$. But $S(\alpha^*, N) \in N \cap A_{\alpha^*}$ and hence $N^* \cap A_{\alpha^*} \neq \phi$.

Now, suppose (2) holds. Let $S : D \to X$ be a net in X which $\mathcal{T}$-converges to x ; let $\mathfrak{A} = \{A : S$ is frequently in $A\}$. Then $x \in \bigcap \{cA : A \in \mathfrak{A}\}$. If $x \notin cA$, then S is eventually in $\mathscr{C}cA$ and hence not frequently in A. By (2), there exists a point y in $\bigcap \{c^*A : A \in \mathfrak{A}\}$. We show now that S is $\mathcal{T}^*$-convergent to y. Let $y \in O^* \in \mathcal{T}^*$ and suppose that S is not eventually in O^*. Then S is frequently in $\mathscr{C}O^*$ and hence $\mathscr{C}O^* \in \mathfrak{A}$. Thus $y \in c^*\mathscr{C}O^* = \mathscr{C}O^*$ and $y \notin O^*$, a contradiction.

2. Invariance of topological properties

In this section we investigate the invariance of several topological properties.

First we show that equivalence is the same as equality for T_1-spaces.

DEFINITION 2.1. For $\mathscr{T}$ a topology on X, we denote by $[\mathscr{T}]$ the set of all topologies $\mathscr{T}^*$ such that $\mathscr{T}\equiv\mathscr{T}^*$.

THEOREM 2.2. *If $\mathscr{T}$ is a T_1-topology on X, then* $[\mathscr{T}]=\{\mathscr{T}\}$.

PROOF. Let $\mathscr{T}^*\equiv\mathscr{T}$: we show that $\mathscr{T}^*=\mathscr{T}$. Firstly, $\mathscr{T}^*$ is a T_1-topology. To see this let $x\neq y$. Then $c(\{x\})\cap c(\{y\})=\phi$ since $\mathscr{T}$ is a T_1-topology and hence by (2) of Theorem 1.5, $c^*(\{x\})\cap c^*(\{y\})=\phi$. Thus, $\mathscr{T}^*$ is a T_1-topology. Let $x\in O\in\mathscr{T}$; then $X=O\cup\mathscr{C}\{x\}$ and by (3) of Theorem 1.5 it follows that $X=\mathrm{Int}^*O\cup\mathrm{Int}^*\mathscr{C}\{x\}$. Hence $x\in\mathrm{Int}^*O\subset O$ and it follows that $\mathscr{T}\subset\mathscr{T}^*$. By symmetry, we get $\mathscr{T}^*\subset\mathscr{T}$.

The converse of Theorem 2.2 is false as seen in

EXAMPLE 2.3. Let $X=\{a,b,c\}$ and $\mathscr{T}=\{\phi, \{a\}, \{a,b\}, \{a,c\}, X\}$. $(X,\mathscr{T})$ is not a T_1-space, but $[\mathscr{T}]=\{\mathscr{T}\}$. Suppose $\mathscr{T}^*\equiv\mathscr{T}$; we must show that $\mathscr{T}^*=\mathscr{T}$. Consider the sequence $S_1 : b,c,b,c,\cdots$. This sequence is not $\mathscr{T}$-convergent and hence is not $\mathscr{T}^*$-convergent. Thus $\mathscr{T}^*\not\equiv\{\phi, X\}$; by Theorem 1.3, each point of X has a $\mathscr{T}^*$-neighborhood other than X. Next consider the sequence $S_2 : a,b,a,b,\cdots$. This sequence is $\mathscr{T}$-convergent to b and hence is $\mathscr{T}^*$-convergent to some point x in X. Let O^* be the smallest $\mathscr{T}^*$-open set containing x. Since $O^*\neq X$ and S_2 is eventually in O^*, it follows that $O^*=\{a,b\}$. Let $S_3 : a,c,a,c,\cdots$. Repeating the above argument, there exists a point y in X such that $\{a,c\}$ is the smallest $\mathscr{T}^*$-open set containing y. Since $\{a\}=\{a,b\}\cap\{a,c\}\in\mathscr{T}^*$, it follows that $x=b$ and $y=c$. Thus $\mathscr{T}^*=\mathscr{T}$.

T_0 is a necessary condition for equivalence to imply equality as shown in

THEOREM 2.4. *Let* $[\mathscr{T}]=\{\mathscr{T}\}$. *Then $\mathscr{T}$ is a T_0-topology.*

PROOF. Suppose that $\mathscr{T}$ is not a T_0-topology; there exist then points $a\neq b$ in X such that $c(\{a\})=c(\{b\})$. Let $\mathscr{T}^*=\mathscr{T}\vee\{\phi, \{b\}, X\}$. It is clear that $\mathscr{T}\subset\mathscr{T}^*$, $\mathscr{T}\neq\mathscr{T}^*$. It suffices to show that $\mathscr{T}\equiv\mathscr{T}^*$. Let $S : D\to X$ be a $\mathscr{T}$-convergent net with limit point x. If $x\neq b$, then S is $\mathscr{T}^*$-convergent to x. If $x=b$, then S is $\mathscr{T}^*$-convergent to a.

The converse of Theorem 2.4 is false as shown in example 1.2 and T_0 is not invariant relative to equivalence. Any separation property stronger than T_1 is invariant (Theorem 2.2). We now investigate the invariance of several other topological properties the definitions of which can be found in [1]. Note that regluarity is not assumed in our definition of paracompactness.

THEOREM 2.5. *The following properties are invariant under equivalence*: (1) *compactness* (2) *sequential compactness* (3) *countable compactness* (4) *connectedness* (5) *Lindelöf* (6) *normality* (7) *collectionwise normality* (8) *full normality* (9) *paracompactness* (10) *countable paracompactness* (11) *metacompactness* (12) *every open cover is even.*

PROOF. The invariance of properties (1) to (3) follows directly from their equivalent formulations in terms of nets. The invariance of (4) is a corollary to the following

LEMMA 2.6. *Let $\mathscr{T}^*\equiv\mathscr{T}$ on X and $A\subset X$. Then A is $\mathscr{T}$-clopen iff A is $\mathscr{T}^*$-clopen.*

PROOF. A is $\mathscr{T}$-clopen iff $cA\cap c\mathscr{C}A=\phi$ iff $c^*A\cap c^*\mathscr{C}A=\phi$ iff A is $\mathscr{T}^*$-clopen. (See (2) of Theorem 1.5.)

To prove (5), let $\mathscr{T}$ be Lindelöf and $\mathscr{T}\equiv\mathscr{T}^*$. If $X=\bigcup\{O^*_\alpha : \alpha\in\Delta\}$ where $O^*_\alpha\in\mathscr{T}^*$ for all α, then by (3) of Theorem 1.5, $X=\bigcup\{\text{Int}O^*_\alpha : \alpha\in\Delta\}$, Int being the $\mathscr{T}$-interior operator. Thus $X=\bigcup\{\text{Int}O^*_{\alpha_i} : i=1,\cdots,n,\cdots\}$ and hence $X=\bigcup\{O^*_{\alpha_i} : i=1,2,\cdots\}$.

To prove (6), let $\mathscr{T}$ be normal and $\mathscr{T}^*\equiv\mathscr{T}$. Let $E^*\cap F^*=\phi$, E^* and F^* being $\mathscr{T}^*$-closed. Then $c^*E^*\cap c^*F^*=\phi$ and by (2) of Theorem 1.5, $cE^*\cap cF^*=\phi$. Since $\mathscr{T}$ is normal, there exist O_1, O_2 in $\mathscr{T}$ such that $cE^*\subset O_1$, $cF^*\subset O_2$ and $O_1\cap O_2=\phi$. Thus $cE^*\cap\mathscr{C}O_1=\phi$ and $cF^*\cap\mathscr{C}O_2=\phi$. Using (2) of Theorem 1.5 again, $c^*E^*\cap c^*\mathscr{C}O_1=\phi$ and $c^*F^*\cap c^*\mathscr{C}O_2=\phi$. Thus $E^*\subset\mathscr{C}c^*\mathscr{C}O_1=\text{Int}^*O_1$ and $F^*\subset\mathscr{C}c^*\mathscr{C}O_2=\text{Int}^*O_2$. Hence E^* and F^* are separated by disjoint, $\mathscr{T}^*$-open sets and $\mathscr{T}^*$ is normal.

Before continuing, we inject the following

LEMMA 2.7. *Let $\mathscr{T}\equiv\mathscr{T}^*$ and $\{A_\alpha : \alpha\in\Delta\}$ a $\mathscr{T}$-locally finite (discrete) family of sets in X. Then $\{A_\alpha : \alpha\in\Delta\}$ is $\mathscr{T}^*$-locally finite (discrete).*

PROOF. For each x in X, there exists an $O_x\in\mathscr{T}$ such that $x\in O_x$ and $\{\alpha : O_x\cap A_\alpha\neq\phi\}$ is finite (is ϕ or a singleton). Thus $X=\bigcup\{O_x : x\in X\}$ and by (3) of Theorem 1.5, $X=\bigcup\{\text{Int}^*O_x : x\in X\}$. That $\{A_\alpha : \alpha\in\Delta\}$ is $\mathscr{T}^*$-locally finite (discrete) follows from the fact that $\text{Int}^*O_x\subset O_x$.

We now prove that (7) is invariant. Let $\mathscr{T}$ be collectionwise normal and $\mathscr{T}^*\equiv\mathscr{T}$. Suppose then that $\{E^*_\alpha : \alpha\in\Delta\}$ is a $\mathscr{T}^*$-discrete family of $\mathscr{T}^*$-closed sets. Then $\{E^*_\alpha : \alpha \in \Delta\}$ is a $\mathscr{T}$-discrete family of sets and hence $\{cE^*_\alpha : \alpha \in \Delta\}$ is also a $\mathscr{T}$-discrete family of closed sets. Thus there exists a disjoint family of $\mathscr{T}$-open sets O_α such that $cE^*_\alpha \cap \mathscr{C}O_\alpha = \phi$ for each α. By (2) of Theorem 1.5, it follows that $c^*E^*_\alpha \cap c^*\mathscr{C}O_\alpha = \phi$ and hence $E^*_\alpha \subset \mathscr{C}c^*\mathscr{C}O_\alpha = \mathrm{Int}^*O_\alpha$. Clearly $\{\mathrm{Int}^*O_\alpha : \alpha \in \Delta\}$ is a disjoint family of $\mathscr{T}^*$-open sets.

Of the remaining properties, we prove only (9) and (12).

To show (9), let $\mathscr{T} \equiv \mathscr{T}^*$ and suppose $\mathscr{T}$ is paracompact. Let $X = \bigcup \{O^*_\alpha : \alpha \in \Delta\}$, $O^*_\alpha \in \mathscr{T}^*$. Then by (3) of Theorem 1.5, $X = \bigcup \{\mathrm{Int}O^*_\alpha : \alpha \in \Delta\}$ and $\mathrm{Int}\, O^*_\alpha \in \mathscr{T}$. There exists then a $\mathscr{T}$-open locally finite refinement $\{O_\gamma : \gamma \in \Gamma\}$ of $\{\mathrm{Int}\, O^*_\alpha : \alpha \in \Delta\}$. By Lemma 2.7 $\{\mathrm{Int}^*O_\gamma : \gamma \in \Gamma\}$ is a $\mathscr{T}^*$-open locally finite refinement of $\{O_\gamma : \gamma \in \Gamma\}$ and hence of $\{O^*_\alpha : \alpha \in \Delta\}$ also. Thus $\mathscr{T}^*$ is paracompact.

Finally, let $\mathscr{T} \equiv \mathscr{T}^*$ and suppose $\mathscr{T}$ has property (12); let $O^*_\alpha : \alpha \in \Delta\}$ be a $\mathscr{T}^*$-open cover of X. By (3) of Theorem 1.5, $\{\mathrm{Int}O^*_\alpha : \alpha \in \Delta\}$ is a $\mathscr{T}$-open cover of X, and hence there exists a $\mathscr{T} \times \mathscr{T}$-open set V containing the diagonal in $X \times X$ such that $\{V[x] : x \in X\}$ refines $\{\mathrm{Int}O^*_\alpha : \alpha \in \Delta\}$. For each $x \in X$, there exists an $O_x \in \mathscr{T}$ such that $(x, x) \in O_x \times O_x \subset V$. Let $W^* = \bigcup \{\mathrm{Int}^*O_x \times \mathrm{Int}^*O_x : x \in X\}$; then W^* is a $\mathscr{T}^* \times \mathscr{T}^*$-open set in $X \times X$ which contains the diagonal and $\{W^*[x] : x \in X\}$ refines $\{O^*_\alpha : \alpha \in \Delta\}$ as the reader can easily verify.

The next result enumerates topological properties which are not invariant relative to equivalence and for which we give counterexamples.

THEORM 2.8. *The following properties are not invariant under equivalence*: (1) R_0($x \in O \in \mathscr{T}$ *implies that* $c\{x\} \subset O$) (2) T_0 (3) *regularity* (4) *complete regularity* (5) *perfect normality* (6) *complete normality* (7) *second axiom of countability* (8) *separability* (9) *local connecteness*

PROOF. We refer to example 1.2 for (1)—(5).

The following example takes care of (7)—(9).

EXAMPLE 2.9. Let X be an uncountable set and $\mathscr{T} = \{\phi, X\}$. Let $x \neq y$ in X and define $\mathscr{T}^* = \{O^* : O^* = X$ or $\{x, y\} \subset \mathscr{C}O^*$ or $x \in O^*$, $y \notin O^*$ and $\mathscr{C}O^*$ is finite$\}$. Since X is the only $\mathscr{T}^*$-open set containing y, it follows from Theorem 1.3 that $\mathscr{T}^* \equiv \mathscr{T}$. $\mathscr{T}$ has properties (7)—(9), but $\mathscr{T}^*$ has none of the properties.

If $X = \{a, b, c, d\}$, $\mathscr{T} = \{\phi, X\}$, $\mathscr{T}^* = \{\phi, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$, then $\mathscr{T} \equiv \mathscr{T}^*$ (Theorem 1.3), $\mathscr{T}$ is completely normal, but $\mathscr{T}^*$ is not.

THEOREM 2.10. *Let $\mathcal{T}^*\equiv\mathcal{T}$ on X and suppose both $\mathcal{T}$ and $\mathcal{T}^*$ are locally connected. If $A\subset X$, then A is a $\mathcal{T}^*$-component iff A is a $\mathcal{T}$-component.*

PROOF. Let A be a $\mathcal{T}$-component of X. Then A is $\mathcal{T}$-clopen and hence A is $\mathcal{T}^*$-clopen by Lemma 2.6. We will show in Theorem 3.2 that $A\cap\mathcal{T}\equiv A\cap\mathcal{T}^*$. Since A is $A\cap\mathcal{T}$-connected, it folows from Theorem 2.5 that A is $A\cap\mathcal{T}^*$-connected. But clopen and connected implies component

3. Properties of equivalent topologies

In this section we further investigate the nature of equivalence. First we note that homeomorphic topologies need not be equivalent.

EXAMPLE 3.1. Let $X=\{a,b,c\}$ and $\mathcal{T}=\{\phi, \{a\}, \{b,c\}, X\}$, $\mathcal{T}^*=\{\phi, \{b\}, \{a,c\}, X\}$. $\mathcal{T}$ and $\mathcal{T}^*$ are homeomorphic, but the sequence $S(2n)=a$ and $S(2n+1)=c$ is $\mathcal{T}^*$-convergent, but not $\mathcal{T}$-convergent.

Secondly we note that subspace topologies generated by equivalent topologies need not be equivalent—in example 2.9, $\mathcal{T}\equiv\mathcal{T}^*$, but $\mathscr{C}\{y\}\cap\mathcal{T}\not\equiv\mathscr{C}\{y\}\cap\mathcal{T}^*$.

Closed subspaces, products and coproducts are more tractable as shown by the following theorems.

THEOREM 3.2. *Let $\mathcal{T}^*\equiv\mathcal{T}$ on X and $Y\subset X$, Y both $\mathcal{T}^*$-closed and $\mathcal{T}$-closed. Then $Y\cap\mathcal{T}^*\equiv Y\cap\mathcal{T}$.*

PROOF. Let $S: D\to Y$ be $Y\cap\mathcal{T}$-convergent. Then $S: D\to X$ is $\mathcal{T}$-convergent and hence $\mathcal{T}^*$-convergent. Since $S[D]\subset Y$ and Y is $\mathcal{T}^*$-closed, $S: D\to Y$ is $Y\cap\mathcal{T}^*$-convergent.

THEOREM 3.3. *Let $(X,\mathcal{T})=\times\{(X_\alpha,\mathcal{T}_\alpha): \alpha\in\Delta\}$ and $(X,\mathcal{T}^*)=\times\{(X_\alpha,\mathcal{T}^*_\alpha): \alpha\in\Delta\}$. Then $\mathcal{T}\equiv\mathcal{T}^*$ iff $\mathcal{T}_\alpha\equiv\mathcal{T}^*_\alpha$ for all $\alpha\in\Delta$. ($X_\alpha\neq\phi$ for all $\alpha\in\Delta$ and $\Delta\neq\phi$.)*

PROOF. Let $\mathcal{T}_\alpha\equiv\mathcal{T}^*_\alpha$ for each $\alpha\in\Delta$ and suppose that $S: D\to X$ is $\mathcal{T}$-convergent. Then $P_\alpha\circ S: D\to X_\alpha$ is $\mathcal{T}$-convergent, P_α denoting the α-projection, and hence $\mathcal{T}^*_\alpha$-convergent for each $\alpha\in\Delta$. It follows then that $S: D\to X$ is $\mathcal{T}^*$-convergent.

Conversely, let $\mathcal{T}\equiv\mathcal{T}^*$ and fix $\beta\in\Delta$. We show that $\mathcal{T}_\beta\equiv\mathcal{T}^*_\beta$: let $S: D\to X_\beta$ be $\mathcal{T}_\beta$-convergent. For $\alpha\neq\beta$, choose $x_\alpha\in X_\alpha$ and define $T: D\to X$ as follows: $P_\beta(T(d))=S(d)$ and $P_\alpha(T(d))=x_\alpha$ for $\alpha\neq\beta$. Then $T: D\to X$ is $\mathcal{T}$-convergent and hence also $\mathcal{T}^*$-convergent. It follows that $S=P_\beta\circ T$ is $\mathcal{T}^*_\beta$-convergent.

THEOREM 3.4. *Let* $(X, \mathscr{T}) = \sum \{(X_\alpha, \mathscr{T}_\alpha) : \alpha \in \Delta\}$ *and* $(X, \mathscr{T}^*) = \sum \{(X_\alpha, \mathscr{T}^*_\alpha) : \alpha \in \Delta\}$, $\{X_\alpha : \alpha \in \Delta\}$ *being a disjoint family of sets. Then* $\mathscr{T} \equiv \mathscr{T}^*$ *iff* $\mathscr{T}_\alpha \equiv \mathscr{T}^*_\alpha$ *for all* $\alpha \in \Delta$.

PROOF. If $\mathscr{T} \equiv \mathscr{T}^*$, then $\mathscr{T}_\alpha \equiv \mathscr{T}^*_\alpha$ for each α by Theorem 3.2 since each X_α is both $\mathscr{T}$-closed and $\mathscr{T}^*$-closed and $\mathscr{T}_\alpha = X_\alpha \cap \mathscr{T}$, $\mathscr{T}^*_\alpha = X_\alpha \cap \mathscr{T}^*$. The converse follows from the fact that X_α is both $\mathscr{T}$-open and $\mathscr{T}^*$-open. We omit the details.

The following theorems deal with the equivalence class determined by a topology. We show that R_0-topologies are the smallest in their equivalence class and that a topology is T_0 if it is maximal in its class.

THEOREM 3.5. *Let* $\mathscr{T}$ *be an* R_0*-topology and* $\mathscr{T} \equiv \mathscr{T}^*$. *Then* $\mathscr{T} \subset \mathscr{T}^*$.

PROOF. Let $x \in O \in \mathscr{T}$. Then $c(x) \subset O$ and hence $c(x) \cap \mathscr{C}O = \phi$. By (2) of Theorem 1.5, $c^*(x) \cap c^* \mathscr{C}O = \phi$ and $x \in \mathscr{C}c^*\mathscr{C}O = \mathrm{Int}^* O \subset O$. Thus $O \in \mathscr{T}^*$.

THEOREM 3.6. *If* $\mathscr{T}$ *is maximal in* $[\mathscr{T}]$, *then* $\mathscr{T}$ *is a* T_0*-topology.*

See the proof of Theorem 2.4. The converse of Theorem 3.6 fails as shown in

EXAMPLE 3.7. Let X be the reals and $\mathscr{T} = \{O : O = \phi$ or $O = X$ or $O = (-\infty, x)$ for some $x \in X\}$; let $\mathscr{T}^* = \mathscr{T} \cup \{(-\infty, O]\}$. Then $\mathscr{T}$ is a T_0-topology, $\mathscr{T} \subset \mathscr{T}^*$, $\mathscr{T} \neq \mathscr{T}^*$, $\mathscr{T} \equiv \mathscr{T}^*$ as the reader can verify.

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REFERENCE

[1] John L. Kelley, *General Topology*, D. Van Nostrand, New York, 1955.