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APPLICATIONS OF HERMITE POLYNOMIALS FOR CERTAIN PROPERTIES OF FOX'S H-FUNCTION OF TWO VARIABLES

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During the course of investigation, the frequent requirements of some generalizations of the mathematical functions which arise in analysis and applied problems–the so-called special functions and their basic interesting properties have been studied by several workers [3, 8, 10, 11, 13, 17 and 18]. The author here evaluates a general integral involving the product of Fox's H-functions of two variables and classical orthogonal Hermite polynomial. This integral plays an important role in the development of certain properties associated with the generalized H-function in two arguments incorporating (i) an expansion theorem (ii) a formal solution of partial differential equation related to a problem of heat conduction (iii) the pure recurrence relation and finally (iv) the summation formulas for the series. These results generalize some known relations due to the writer [19, 20] and extend many interesting results on H-and G-functions and other functions appearing in Applied Mathematics and Mathematical Physics as particular cases by appropriately specializing the parameters. Relations on the double-integral-expansion analogues have been also illustrated.

1. Introduction

A generalization of Sharma's S-function of two variables [17] has been recently introduced by Munot and Kalla [11] in the form

$$H\left[\begin{array}{c|c|c} \begin{bmatrix} m_1, & 0 \\ p_1-m_1, & q_1 \end{bmatrix} & (a_{p_1}, A_{p_1}) ; (b_{q_1}, B_{q_1}) & \\ \begin{pmatrix} m_2, & n_2 \\ p_2-m_2, & q_2-n_2 \end{pmatrix} & (c_{p_2}, C_{p_2}) : (d_{q_2}, D_{q_2}) & x, y \\ \begin{pmatrix} m_3, & n_3 \\ p_3-m_3, & q_3-n_3 \end{pmatrix} & (e_{p_3}, E_{p_3}) ; (f_{q_3}, F_{q_3}) & \end{array}\right] \tag{1.1}$$

$$=\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2} F(\xi+\eta)\Phi(\xi,\eta)x^{\xi}y^{\eta}\,d\xi\,d\eta,$$

where

$$F(\xi+\eta)=\frac{\prod_{j=1}^{m_1}\Gamma(a_j+A_j\xi+A_j\eta)}{\prod_{j=m_1+1}^{p_1}\Gamma(1-a_j-A_j\xi-A_j\eta)\prod_{j=1}^{q_1}\Gamma(b_j+B_j\xi+B_j\eta)},$$

$$\Phi(\xi,\eta)=\frac{\prod_{j=1}^{m_2}\Gamma(1-c_j+C_j\xi)\prod_{j=1}^{n_2}\Gamma(d_j-D_j\xi)\prod_{j=1}^{m_3}\Gamma(1-e_j+E_j\eta)\prod_{j=1}^{n_3}\Gamma(f_j-F_j\eta)}{\prod_{j=m_2+1}^{p_2}\Gamma(c_j-C_j\xi)\prod_{j=n_2+1}^{q_2}\Gamma(1-d_j+D_j\xi)\prod_{j=m_3+1}^{p_3}\Gamma(e_j-E_j\eta)\prod_{j=n_3+1}^{q_3}\Gamma(1-f_j+F_j\eta)}.$$

The contours L_1, L_2, notations and convergence of (1.1) etc. are described in [11] and we omit such details.

Hermite polynomials

Hermite polynomials are classical orthogonal polynomials associated with the interval $(-\infty, \infty)$ and an exponential weight function. Sneddon [16, p. 150] has defined the Hermite polynomials $H_n(x)$ for integral values of n and all real values of x by the identity

$$\exp\,(2xt-t^2)=\sum_{n=0}^{\infty}\frac{H_n(x)}{n\,!}\,t^n. \tag{1.2}$$

In this paper we shall adopt Szegö's [15] notation and regard Hermite polynomials, $H_n(x)$, as the suitably standardized orthogonal polynomials associated with

$$a=-\infty,\ b=\infty,\ w(x)=\exp(-x^2),\ x=1. \tag{1.3}$$

There exists a considerable body of information on the subject of generalizations and unified development for many of the mathematical functions which arise in applied problems, as well as the attendant mathematical theory for their approximations. An attempt has been made by different researchers [3, 8, 10, 11, 13 & 18] to unify and to extend certain results on generalized functions scattered throughout literature.

In this paper the author derives a more general integral for the product of Fox's H-function of two variables and Hermite polynomial. This integral is extensively used to establish many results involving generalized H-functions of two variables leading to (i) an expansion theorem (ii) a formal solution of a problem of heat conduction (iii) a pure recurrence relation and lastly (iv) the summation formulas of the series. These results are of general character in the branch of special-functions.

It is observed that some recent formulas scattered in special functions admit themselves of interesting extensions which provide one with the unification of several results. Also, analogues of the double-integral-expansion for generalized H-functions of two variables are exhibited.

2. Notations and preliminary results

Often, as a space saver, we use the notations:

(i) the L.H.S. of (1.1) shall be written as $H[x,y]$;

(ii) the symbol (a_p, A_p) is used to denote the set of p ordered pairs (a_1, A_1), (a_2, A_2), $\cdots$, (a_p, A_p) ;

(iii) $a_p(b_q)$ stands for $p(q)$ parameters $a_1, \cdots, a_p$ $(b_1, \cdots, b_q)$;

(iv) $$f=f\binom{m,\ n}{p,\ q}=\begin{cases}\begin{bmatrix} m_1+2, & 0 \\ p_1-m_1, & q_1+1\end{bmatrix} \\ \begin{pmatrix} m_2, & n_2 \\ p_2-m_2, & q_2-n_2\end{pmatrix} \\ \begin{pmatrix} m_3, & n_3 \\ p_3-m_3, & q_3-n_3\end{pmatrix},\end{cases}$$

(v) $$f_1=f_1\binom{m,\ n}{p,\ q}=\begin{cases}\begin{bmatrix} m_1+1, & 0 \\ p_1-m_1, & q_1+1\end{bmatrix} \\ \begin{pmatrix} m_2, & n_2 \\ p_2-m_2, & q_2-n_2\end{pmatrix} \\ \begin{pmatrix} m_3, & n_3 \\ p_3-m_3, & q_3-n_3\end{pmatrix},\end{cases}$$

(vi) $$g=g(p,\ q)=\begin{cases}(c_{p_2},\ C_{p_2}) ; (d_{q_2},\ D_{q_2}) \\ (e_{p_3},\ E_{p_3}) ; (f_{q_3},\ F_{q_3}).\end{cases}$$

The integral (1.1) converges under the following sets of conditions:

$$(2.1)\quad \begin{cases}\lambda_1\equiv\sum_1^{p_1}A_j+\sum_1^{p_2}C_j-\sum_1^{q_1}B_j-\sum_1^{q_2}D_j\leq 0 ; \mu_1\equiv\sum_1^{p_1}A_j+\sum_1^{p_3}E_j-\sum_1^{q_1}B_j-\sum_1^{q_3}F_j\leq 0, \\ \lambda_2=\sum_1^{m_1}A_j-\sum_{m_1+1}^{p_1}A_j-\sum_1^{q_1}B_j+\sum_1^{m_2}C_j-\sum_{m_2+1}^{p_2}C_j+\sum_1^{n_2}D_j-\sum_{n_2+1}^{q_2}D_j>0, |\arg x|<\frac{1}{2}\lambda_2\pi, \\ \mu_2=\sum_1^{m_1}A_j-\sum_{m_1+1}^{p_1}A_j-\sum_1^{q_1}B_j+\sum_1^{m_3}E_j-\sum_{m_3+1}^{p_3}E_j+\sum_1^{n_3}F_j-\sum_{n_3+1}^{q_3}F_j>0, |\arg y|<\frac{1}{2}\mu_2\pi.\end{cases}$$

$$
(2.2)\quad \begin{cases}
\theta_1=\theta_1(l)=(l+1,\sigma),\ \left(l+\frac{1}{2},\sigma\right);\ \theta_2=\theta_2(l,k)=(l-k+1,\sigma),\\
\theta_3=\theta_3(l,r)=(l-r+1,\sigma),\ \theta_4=\theta_4(l,k)=\left(l-\frac{1}{2}k+1,\sigma\right),\\
\theta_5=\theta_5(l,r)=\left(l-\frac{1}{2}r+1,\sigma\right),\ \theta_6=\theta_6(l,k)=\left(l-k+\frac{1}{2},\sigma\right),\\
\theta_7=\theta_7(l,k)=\left(l-k+\frac{3}{2},\sigma\right),\ \theta_8=\theta_8(l)=\left(l+\frac{3}{2},\sigma\right),\ (l+1,\sigma),\\
\theta_9=\theta_9(l,k,s)=\left(l+k-s+\frac{1}{2},\sigma\right),\ \theta_{10}=\theta_{10}(l,\rho)=\left(l+\frac{1}{2}\rho+\frac{1}{2},\sigma\right).
\end{cases}
$$

In the investigation of the present work we make use of the following results:

(a) Integrals involving Hermite polynomials [12, p. 115, (3) and (4)] :

$$(2.3)\quad \int_{-\infty}^{\infty} e^{-x^2}\, x^{2l}\, H_{2n}(x)\, dx=\frac{2^{2n}\Gamma(l+1)\Gamma\left(l+\frac{1}{2}\right)}{\Gamma(l-n+1)},\quad 0\leq n\leq l.$$

$$(2.4)\quad \int_{-\infty}^{\infty} e^{-x^2} x^{2l+1}\, H_{2n+1}(x)\, dx=\frac{2^{2n+1}\Gamma(l+1)\Gamma\left(l+\frac{3}{2}\right)}{\Gamma(l-n+1)},\quad 0\leq n\leq l.$$

(b) Orthogonality-property of the Hermite polynomials [14, p. 192, (5)—(6)] :

$$(2.5)\quad \int_{-\infty}^{\infty} \exp(-x^2)\, H_n(x)\, H_m(x)\, dx=h_n\, \delta_{mn},$$

where $h_n=2^n n!\sqrt{\pi}$, $\delta_{mn}=0$ for $m\neq n$ and $\delta_{mn}=1$ for $m=n$.

(c) Pure recurrence relation for Hermite polynomials [6, p. 193, (10)] :

$$(2.6)\quad H_{n+1}(x)-2xH_n(x)+2nH_{n-1}(x)=0:$$

(d) Definition of the Hermite polynomials [14, p. 197] :

$$(2.7)\quad H_{2k}(x)=\sum_{s=0}^{k}\frac{(-1)^s\,(2k)!\,(2k)^{2k-2s}}{s!\,(2k-2s)!},$$

(e) Series of Hermite polynomials [6, p. 216, (28)] :

$$(2.8)\quad |x|^{\rho}=\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\rho\right)}{\sqrt{\pi}}\sum_{m=0}^{\infty}(-1)^m\frac{\left(-\frac{1}{2}\rho\right)_m}{(2m)!}H_{2m}(x),\quad \rho>-1.$$

(f) Formulae [14, p. 32, Ex. 8, p. 197] :

$$(2.9)\quad (\alpha)_{n-k}=\frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k}\ \text{for}\ 0\leq k\leq n:\ (2k)!=2^{2k}k!\left(\frac{1}{2}\right)_k.$$

(g) Relations between Hermite and Laguerre polynomials [6, p. 193, (2) and (3)] :

$$(2.10)\quad H_{2m}(x)=(-1)^m 2^{2m} m!\, L_m^{-\frac{1}{2}}(x^2),$$

$$(2.11)\quad H_{2m+1}(x)=(-1)^m\, 2^{2m+1}\, m!\, x\, L_m^{\frac{1}{2}}(x^2).$$

3. The general integral

Firstly, we evaluate an integral involving the product of Fox's H-function in two arguments and Hermite polynomial which is based on the principle of interchanging the order of integration and then term-by-term integration by invoking the known formula. Later on this integral is employed to study for a variety of several interesting properties concerned with the generalized functions.

We begin by considering the integral

$$I=\int_{-\infty}^{\infty} e^{-u^2}\, u^{2l}\, H_{2k}(u)\, H\,[xu^{2\sigma},\ yu^{2\sigma}]\, du$$

where σ is a positive integer >0 and $0\leq k\leq l$.

On substituting the contour integral (1.1) for $H\,[xu^{2\sigma},\ yu^{2\sigma}]$, if we invert the order of integration which is easily seen to be justified due to the absolute convergence of the integrals involved in the process and by an application of de la Vallée Poussin's theorem [1, p. 504] in view of conditions stated in (2.1) earlier, and then interpret the inner u-integral with the help of (2.3), we obtain

$$2^{2k}\frac{1}{(2\pi i)^2}\int_{L_1}\int_{L_2} F(\xi+\eta)\Phi(\xi,\eta)\ \frac{\Gamma(l+\sigma\xi+\sigma\eta+1)\Gamma\left(l+\sigma\xi+\sigma\eta+\frac{1}{2}\right)}{\Gamma(l+\sigma\xi+\sigma\eta-k+1)}x^{\xi}y^{\eta}d_{\xi}d_{\eta}.$$

On applying (1.1), we get the basic result

$$(3.1)\quad I=2^{2k}\ H\left[f\left|\begin{matrix}\theta_1,\ (a_{p_1},\ A_{p_1})\ ;\ (b_{q_1},\ B_{q_1}),\ \theta_2\\ g\end{matrix}\right|\ x,y\right]$$

provided $\sigma>0$, $0\leq k\leq l$, $p_1\geq m_1\geq 0$, $p_2\geq m_2\geq 0$, $q_1\geq 0$, $q_2\geq n_2\geq 0$, $p_3\geq m_3\geq 0$, $q_3\geq n_3\geq 0$, $\lambda_1\leq 0$, $\mu_1\leq 0$, $\lambda_2>0$, $\mu_2>0$, $\mathrm{Re}\left(\sigma\frac{d_i}{D_i}+\sigma\frac{f_j}{F_j}\right)>0$ $(1\leq i\leq n_2;\ 1\leq j\leq n_3)$ where λ_1, λ_2, μ_1, μ_2, θ_1 and θ_2 have the same meaning as in (2.1) and (2.2).

4. An expansion theorem

The expansion of a given, "arbitrary" or analytic function in a series or orthogonal polynomials has been discussed often and in great detail by Szegö [15]. Orthogonal polynomials occupy a prominent place in numerical integration and also in the determination of coefficients. Since the $H_n(x)$ form an orthogonal set, the classical technique for expanding the polynomials is available in the general theory.

THEOREM:

(4.1) *If σ is a positive integer >0, $-\infty<u<\infty$, then*

$$u^{2l}H[xu^{2\sigma}, yu^{2\sigma}]=\frac{1}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{1}{(2r)!}H\left[f\left|\begin{matrix}\theta_1,(a_{p_1},A_{p_1}):(b_{q_1},B_{q_1}),\theta_3\\ g\end{matrix}\right|x,y\right]H_{2r}(u)$$

provided $p_1\geq m_1\geq 0$, $p_2\geq m_2\geq 0$, $q_1\geq 0$, $q_2\geq n_2\geq 0$, $p_3\geq m_3\geq 0$, $q_3\geq n_3\geq 0$, $\lambda_1\leq 0$, $\mu_1\leq 0$, $\lambda_2>0$, $\mu_2>0$, $\mathrm{Re}\left(\sigma\frac{d_i}{D_i}+\sigma\frac{f_j}{F_j}\right)>0\,(1\leq i\leq n_2\,;\,1\leq j\leq n_3)$ *where* $\lambda_1,\lambda_2,\mu_1,\mu_2,\theta_1$ *and* θ_3 *are the same as given in* (2.1) *and* (2.2).

PROOF. Suppose formally that

$$(4.2)\quad f(u)=u^{2l}\,H[xu^{2\sigma}, yu^{2\sigma}]=\sum_{r=0}^{\infty}M_r H_{2r}(u),\quad (-\infty<u<\infty).$$

From (4.2) we obtain M_r in a purely formal manner. With that value of M_r we have then assumed that the series on the right in (4.2) actually converges to $f(u)$, providing $f(u)$ is sufficiently well behaved.

Multiply both sides by $e^{-u^2}H_{2k}(u)$ and integrate from $-\infty$ to ∞. Then in view of (4.2) it follows formally that

$$(4.3)\quad \int_{-\infty}^{\infty}e^{-u^2}\,u^{2l}H_{2k}(u)H[xu^{2\sigma}, yu^{2\sigma}]\,du$$
$$=\sum_{r=0}^{\infty}M_r\int_{-\infty}^{\infty}e^{-u^2}\,H_{2k}(u)\,H_{2r}(u)\,du.$$

All the integrals on the right in (4.3) vanish except for the single term for which $r=k$.

Now use (3.1) and (2.5) in (4.3) to obtain

$$(4.4)\quad M_k=\frac{1}{(2k)!\sqrt{\pi}}\,H\left[f\left|\begin{matrix}\theta_1,\ (a_{p_1},A_{p_1}):(b_{q_1},B_{q_1}),\ \theta_2\\ g\end{matrix}\right|x,y\right].$$

Finally, apply (4.4) in (4.2), the result (4.1) follows.

5. Heat conduction problem

A genius effort has been performed by Kampé de Fériet J. [9, p. 193] to utilize the Hermite polynomials in solving a heat conduction equation. Recently, with the aid of these polynomials, Bhonsle [2, p. 360] has obtained a formal solution of partial differential equation

$$(5.1)\qquad \frac{\partial V}{\partial t}=h\frac{\partial^2 V}{\partial u^2}-hvu^2$$

under certain prescribed boundary conditions which arise from the physical situation.

Here our problem is to obtain a function $v(u,t)$ which satisfied the partial differential equation (5.1) in view of initial and boundary conditions.

Following Churchill [4, p. 130], (5.1) can be associated with a heat conduction equation

$$(5.2) \qquad \frac{\partial V}{\partial t}=h\frac{\partial^2 V}{\partial u^2}-s(V-V_0)$$

subject to $V_0=0$ and $s=hu^2$.

When $v(u,t)$ tends to zero for sufficiently large values of t and if $|u|\to\infty$, Bhonsle [2, p. 360, (2.3)] has established the solution of (5.1) as

$$(5.3) \qquad V(u,\ t)=\sum_{r=0}^{\infty} P_r\ e^{-(1+2r)ht}\ H_r(u).$$

When $t=0$, let $V(u,\ 0)=f(u)$.

If $f(u)=u^{2l}H\,[xu^{2\sigma},\ yu^{2\sigma}]$, then we suppose formally that

$$(5.4) \qquad f(u)=u^{2l}\ H\,[xu^{2\sigma},\ yu^{2\sigma}]=\sum_{r=0}^{\infty} P_r\, H_r(u).$$

The idea of obtaining P_r is similar to that for M_r and we omit details.

Therefore, employ (2.5) and (3.1) in (5.4), it follows formally that

$$(5.5) \qquad P_k=\frac{1}{k!\sqrt{\pi}}\ H\left[f\left|\begin{matrix}\theta_1,\ (a_{p_1},\ A_{p_1})\ ;\ (b_{q_1},\ B_{q_1}),\ \theta_4\\ g\end{matrix}\right|\ x,\ y\right].$$

Now, from (5.5) and (5.3), the solution is equal to

$$(5.6) \quad V(u,\ t)=\frac{1}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{e^{-(1+2r)ht}}{r!}\ H\left[f\left|\begin{matrix}\theta_1,(a_{p_1},A_{p_1})\ ;\ (b_{q_1},B_{q_1}),\theta_5\\ g\end{matrix}\right|\ x,y\right]H_r(u)$$

valid for $\sigma>0$, $-\infty<u<\infty$, $p_1\geq m_1\geq 0$, $p_2\geq m_2\geq 0$, $q_1\geq 0$, $q_2\geq n_2\geq 0$, $p_3\geq n_3\geq 0$, $q_3\geq n_3\geq 0$, $\lambda_1\leq 0$, $\mu_1\leq 0$, $\lambda_2>0, \mu_2>0$, $\mathrm{Re}\left(\sigma\frac{d_i}{D_i}+\sigma\frac{f_j}{F_j}\right)>0$ $(1\leq i\leq n_2\ ;\ 1\leq j\leq n_3)$

where λ_1, λ_2, μ_1, μ_2, θ_1 and θ_5 are defined in (2.1) and (2.2).

6. The pure recurrence relation

In this section we present a pure recurrence formula for generalized Fox's H-functions of two variables which has been recorded with the help of a known recurrence relation for Hermite polynomials and (3.1).

The Basic Relation:

$$(6.1) \qquad H\left[f\left|\begin{matrix}\theta_1,\ (a_{p_1},\ A_{p_1})\ ;\ (b_{q_1},\ B_{q_1}),\ \theta_6\\ g\end{matrix}\right|x,\ y\right]$$
$$+kH\left[f\left|\begin{matrix}\theta_1,\ (a_{p_1},\ A_{p_1})\ ;\ (b_{q_1},\ B_{q_1}),\ \theta_7\\ g\end{matrix}\right|x,\ y\right]$$

$$=H\left[f\left|\begin{matrix}\theta_8, (a_{p_1}, A_{p_1}) : (b_{q_1}, B_{q_1}), \theta_7\\ g\end{matrix}\right| x, y\right]$$

where σ is a positive integer >0 and conditions of validity are:
$p_1 \geq m_1 \geq 0$, $p_2 \geq m_2 \geq 0$, $q_2 \geq n_2 \geq 0$, $p_3 \geq m_3 \geq 0$, $q_3 \geq n_3 \geq 0$, $q_1 \geq 0$, $\lambda_1 \leq 0$, $\mu_1 \leq 0$, $\lambda_2 > 0$, $\mu_2 > 0$, $\mathrm{Re}\left(\sigma \frac{d_i}{D_i} + \sigma \frac{f_j}{F_j}\right) > 0$, $(1 \leq i \leq n_2 : 1 \leq j \leq n_3)$ and λ_1, λ_2, μ_1, μ_2, θ_1, θ_6, θ_7 and θ_8 are the same as given in (2.1) and (2.2).

PROOF. From (2.6), we have

$$H_{2k+1}(u) + 4k\, H_{2k-1}(u) = 2u\, H_{2k}(u). \tag{6.12}$$

We may multiply both members by $e^{-u^2}\, u^{2l}\, H[xu^{2\sigma}, yu^{2\sigma}]$ and then integrate with respect to u over $(-\infty, \infty)$ to obtain

$$\begin{aligned}(6.3)\quad &\int_{-\infty}^{\infty} e^{-u^2}\, u^{2l}\, H_{2k+1}(u) H\,[xu^{2\sigma}, yu^{2\sigma}]\, du \\ &+4k \int_{-\infty}^{\infty} e^{-u^2}\, u^{2l}\, H_{2k-1}(u) H\,[xu^{2\sigma}, yu^{2\sigma}]\, du \\ &= 2\int_{-\infty}^{\infty} e^{-u^2}\, u^{2l+1}\, H_{2k}(u) H\,[xu^{2\sigma}, yu^{2\sigma}]\, du\end{aligned}$$

provided that the integrals involved in (6.3) exist.

Using (3.1) with proper changes in parameters etc. for evaluation of integrals in (6.3), then after simplifications, we arrive at (6.1).

7. Summation formulas

Here we are concerned with two summation formulae for the series involving Fox's H-functions of two variables. The first formula is established from the definition of the Hermite polynomials and using (3.1). While the second is formulated from the expansion of u^{ρ} in a series of Hermite polynomials and applying (3.1).

The First Summation Formula:

$$H\left[f\left|\begin{matrix}\theta_1, (a_{p_1}, A_{p_1}) : (b_{q_1}, B_{q_1}), \theta_2\\ g\end{matrix}\right| x, y\right] \tag{7.1}$$

$$= \sum_{s=0}^{k} \frac{(-1)^s(-2k)_{2s}}{s!\, 2^{2s}} H\left[f_1\left|\begin{matrix}\theta_9, (a_{p_1}, A_{p_1}) : (b_{q_1}, b'_{q_1})\\ g\end{matrix}\right| x, y\right]$$

valid for $\sigma > 0$, $\lambda_1 \leq 0$, $\lambda_2 > 0$, $\mu_1 \leq 0$, $\mu_2 > 0$, $p_1 \geq m_1 \geq 0$, $p_2 \geq m_2 \geq 0$, $q_1 \geq 0$, $q_2 \geq n_2 \geq 0$,

$p_3 \geq m_3 \geq 0$, $q_3 \geq n_3 \geq 0$, $\mathrm{Re}\left(\sigma \frac{d_i}{D_i} + \sigma \frac{f_j}{F_j}\right) > 0$ $(1 \leq i \leq n_2,\ 1 \leq i \leq n_2,\ 1 \leq j \leq n_3)$

where λ_1, λ_2, μ_1, μ_2, θ_1, θ_2 and θ_9 have the same values as referred to earlier.

The Second Summation Formula:

If σ is a positive integer $p_1 \geq m_1 \geq 0$, $p_2 \geq m_2 \geq 0$, $q_1 \geq 0$, $q_2 \geq n_2 \geq 0$, $p_3 \geq m_3 \geq 0$, $q_3 \geq n_3 \geq 0$, then

$$(7.2) \quad H\left[f_1 \left| \begin{matrix} \theta_{10}, (a_{p_1}, A_{p_1}) ; (b_{q_1}, B_{q_1}) \\ g \end{matrix} \right| x, y\right]$$

$$= \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\rho\right)}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\rho\right)_k}{k! \left(\frac{1}{2}\right)_k} H\left[f \left| \begin{matrix} \theta_1, (a_{p_1}, A_{p_1}) ; (b_{q_1}, B_{q_1}), \theta_2 \\ g \end{matrix} \right| x, y\right],$$

provided $\rho > -1$, $\lambda_1 \leq 0$, $\lambda_2 > 0$, $\mu_1 \leq 0$, $\mu_2 > 0$, $\mathrm{Re}\left(\sigma \frac{d_i}{D_i} + \sigma \frac{f_j}{F_j}\right) > 0$ $(1 \leq i \leq n_2$; $1 \leq j \leq n_3)$ where λ_1, λ_2, μ_1, μ_2, θ_1, θ_2 and θ_{10} have the same meaning as referred to earlier.

PROOF. (a) To prove (7.1), we can use (2.9) in (2.7) to get

$$(7.3) \qquad H_{2k}(u) = \sum_{s=0}^{k} \frac{(-1)^s (-2k)_{2s} (2u)^{2k-2s}}{s!}.$$

Multiply both sides of (7.3) by $e^{-u^2} u^{2l} H[xu^{2\sigma}, yu^{2\sigma}]$ and integrate with respect to u from $-\infty$ and ∞, then change the order of integration and summation in the right hand side which can readily be justified by Bromwich [1, p. 500] under the conditions stated in (7.1), we obtain

$$(7.4) \qquad \int_{-\infty}^{\infty} e^{-u^2} u^{2l} H_{2k}(u) H[xu^{2\sigma}, yu^{2\sigma}]\, du$$

$$= \sum_{s=0}^{k} \frac{(-1)^s (-2k)_{2s}\, 2^{2k-2s}}{s!} \int_{-\infty}^{\infty} e^{-u^2} u^{2l+2k-2s} H[xu^{2\sigma}, yu^{2\sigma}]\, du.$$

Now apply (3.1) with suitable adjustments of parameters etc. to evaluate the integrate involved there in (7.4). This completes the proof of (7.1).

(b) To establish (7.2), we start from (2.8) and (2.9) to record

$$(7.5) \qquad u^{\rho} = \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\rho\right)}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\rho\right)_k}{2^{2k}\, k! \left(\frac{1}{2}\right)_k} H_{2k}(u), \quad \rho > -1.$$

Proceeding as above, we have

$$(7.6) \quad \int_{-\infty}^{\infty} e^{-u^2} u^{2l+\rho} H[xu^{2\sigma}, yu^{2\sigma}]\, du$$

$$= \frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\rho\right)}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \left(-\frac{1}{2}\rho\right)_k}{2^{2k} k! \left(\frac{1}{2}\right)_k} \int_{-\infty}^{\infty} e^{-u^2} u^{2l} H_{2k}(u) H[xu^{2\sigma}, yu^{2\sigma}]\, du.$$

Evaluation of integrals in (7.6) is similar to that for (7.4) and we omit details. Hence the formula (7.2) is completely proved.

8. Double-integral-expansion analogous

Here we consider the double-integral-expansion relations analogous to (3.1) and (4.1). We state results and omit details as the proofs are much akin to those developed in sections 3 and 4. Integral:

$$(8.1)\qquad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(u^2+v^2)}\, u^{2l}\, v^{2m}\, H_{2k}(u)H_{2h}(v)H\,[xu^{2\sigma},\ yv^{2\rho}]\,dudv$$

$$=2^{2k+2h}\, H\left[\begin{matrix}\begin{bmatrix} m_1, & 0 \\ p_1-m_1, & q_1\end{bmatrix}\\ \begin{pmatrix} m_2+2, & n_2 \\ p_2-m_2, & q_2-n_2+1\end{pmatrix}\\ \begin{pmatrix} m_3+2, & n_3 \\ p_3-m_3, & q_3-n_3+1\end{pmatrix}\end{matrix}\ \middle|\ \begin{matrix}(a_{p_1},\ A_{p_1}) : (b_{q_1},\ B_{q_1})\\ (-l,\sigma),\ \left(-l+\frac{1}{2},\sigma\right),\ (d_{q_2}, D_{q_2}),\\ (c_{p_2},\ C_{p_2}) : (-l+k,\ \sigma)\\ (-m,\rho),\ \left(-m+\frac{1}{2},\rho\right), (f_{q_3},\ F_{q_3}),\\ (e_{p_3},\ E_{p_3}) : (-m+h,\ \rho)\end{matrix}\ \middle|\ x,\ y\right].$$

Expansion:

$$(8.2)\qquad u^{2l}\ v^{2m}\ H\,[xu^{2\sigma},\ yv^{2\rho}]$$

$$=\frac{1}{\pi}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{1}{(2r)!\ (2s)!}\ H_{2r}(u)\ H_{2s}(v)$$

$$H\left[\begin{matrix}\begin{bmatrix} m_1\ , & 0 \\ p_1-m_1, & q_1\end{bmatrix}\\ \begin{pmatrix} m_2+2, & n_2 \\ p_2-m_2, & q_2-n_2+1\end{pmatrix}\\ \begin{pmatrix} m_3+2, & n_3 \\ p_3-m_3, & q_3-n_3+1\end{pmatrix}\end{matrix}\ \middle|\ \begin{matrix}(a_{p_1},\ A_{p_1}) : (b_{q_1}, B_{q_1})\\ (-l,\ \sigma),\ \left(-l+\frac{1}{2},\ \sigma\right), (c_{p_2}, C_{p_2}) : (d_{q_2}, D_{q_2}), (-l+r, \sigma)\\ (-m,\ \rho),\ \left(-m+\frac{1}{2},\ \rho\right),\ (e_{p_3},\ E_{p_3}) : (f_{q_3},\ F_{q_3})\end{matrix}\ \middle|\ x, y\right]$$

where the conditions of validity for these situations are enumerated below:

$\sigma>0$, $\rho>0$, $0\leq k\leq l$, $0\leq h\leq m$, $-\infty<u<\infty$, $-\infty<v<\infty$, $p_1\geq m_1\geq 0$, $q_1\geq 0$, $p_2\geq m_2\geq 0$, $q_2\geq n_2\geq 0$, $p_3\geq m_3\geq 0$, $q_3\geq n_3\geq 0$, $\lambda_1\leq 0$, $\mu_1\leq 0$, $\lambda_2>0$, $\mu_2>0$, $\mathrm{Re}\left(\sigma\frac{d_i}{D_i}\right)>0$, $(1\leq i\leq n_2)$, $\mathrm{Re}\left(\rho\ \frac{f_j}{F_j}\right)>0$, $(1\leq j\leq n_3)$ and λ_1, λ_2, μ_1 and μ_2 have the same meaning as in (2.1) and (2.2).

9. Particular cases

Various generalizations, proved above, will give rise to many useful results involving the functions of mathematical physics, some of which are known and others are believed to be new, as particular cases, by making suitable substitutions. However, we illustrate below some of the very interesting cases.

(a) If we set $A_j=B_i=\cdots$etc.$=1(1\leq j\leq p_1,\ 1\leq i\leq q_1,\ \cdots$etc.) and on specialization of the parameters etc., the results (3.1), (4.1), (5.6) and (6.1) yield the known results [19, 20] on generalized S-functions of two variables which, in turn, lead to generalizations of some recent results [19] on product of two Meijer's G-functions when $A=B=0$. Similary (7.1)—(7.2) will have the corresponding relations.

(b) In (3.1) and (4.1), if we substitute $A=B=0$, we can obtain the results associated with product of Fox's H-functions [7, p. 408] of one variable, which are generalizations of Meijer's G-functions [5, p. 207, (1)] unifying a large number of special functions [5, pp. 215—222].

(c) In the main results, if we put $p_1=m_1$, $p_3=m_3$, $n_3=1$, $f_1=0$ and replace p_1+m_2, p_1+p_2, q_1+q_2 and n_2 by m, p, q and n respectively, then by proper choice of parameters etc., and take the limit as $y\longrightarrow 0$, we obtain the interesting cases presenting Fox's H-functions as

(i) Integral:

$$(9.1)\quad \int_{-\infty}^{\infty} e^{-u^2} u^{2l} H_{2k}(u)\, H_{p,q}^{n,m}\left[xu^{2\sigma}\,\middle|\,\begin{matrix}(a_p,\ A_p)\\(b_q,\ B_q)\end{matrix}\right] du$$

$$=2^{2k} H_{p+2,q+1}^{n,m+2}\left[x\,\middle|\,\begin{matrix}(-l,\sigma),\ \left(-l+\frac{1}{2},\sigma\right),\ (a_p,A_p)\\(b_q,B_q),\ (-l+k,\sigma)\end{matrix}\right].$$

(ii) Expansion-formula:

$$(9.2)\quad u^{2l} H_{p,q}^{n,m}\left[xu^{2\sigma}\,\middle|\,\begin{matrix}(a_p,A_p)\\(b_q,\ B_q)\end{matrix}\right]$$

$$=\frac{1}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{1}{(2r)!}\,H_{p+2,q+1}^{n,m+2}\left[x\,\middle|\,\begin{matrix}(-l,\sigma),\ \left(-l+\frac{1}{2},\sigma\right),(a_p,A_p)\\(b_q,B_q),\ (-l+r,\sigma)\end{matrix}\right] H_{2r}(u).$$

(iii) Solution of the problem related to heat conduction equation:

(9.3) $v(u,t)$

$$=\frac{1}{\sqrt{\pi}}\sum_{r=0}^{\infty}\frac{e^{-(1+2r)ht}}{r!}\,H_{p+2,q+1}^{n,m+2}\left[x\,\middle|\,\begin{matrix}(-l,\sigma),\ \left(-l+\frac{1}{2},\sigma\right),(a_p,A_p)\\(b_q,B_q),\ \left(-l+\frac{1}{2}r,\sigma\right)\end{matrix}\right] H_r(u).$$

(iv) Pure recurrence relation:

$$(9.4)\qquad H^{n,m+2}_{p+2,q+1}\left[x\left|\begin{matrix}(-l,\sigma),\ \left(-l+\frac{1}{2},\sigma\right),\ (a_p,A_p)\\ (b_q,B_q),\ \left(-l+k+\frac{1}{2},\sigma\right)\end{matrix}\right.\right].$$

$$+k\ H^{n,m+2}_{p+2,q+1}\left[x\left|\begin{matrix}(-l,\sigma),\ \left(-l+\frac{1}{2},\sigma\right),\ (a_p,A_p)\\ (b_q,B_q),\ \left(-l+k-\frac{1}{2},\sigma\right)\end{matrix}\right.\right]$$

$$=H^{n,m+2}_{p+2,q+1}\left[x\left|\begin{matrix}\left(-l-\frac{1}{2},\sigma\right),\ (-l,\sigma),\ (a_p,A_p)\\ (b_q,B_q),\ \left(-l+k-\frac{1}{2},\sigma\right)\end{matrix}\right.\right].$$

(v) Summation formulae:

$$(9\cdot5)\qquad H^{n,m+2}_{p+2,q+1}\left[x\left|\begin{matrix}(-l,\sigma),\ \left(-l+\frac{1}{2},\sigma\right),\ (a_p,A_p)\\ (b_q,B_q),\ (-l+k,\sigma)\end{matrix}\right.\right]$$

$$=\sum_{s=0}^{k}\frac{(-1)^s(-2k)_{2s}}{s!\ 2^{2s}}\ H^{n,m+1}_{p+1,q}\left[x\left|\begin{matrix}\left(-l-k+s+\frac{1}{2},\sigma\right),\ (a_p,A_p)\\ (b_q,B_q)\end{matrix}\right.\right]$$

and

$$(9\cdot6)\qquad H^{n,m+1}_{p+1,q}\left[x\left|\begin{matrix}\left(-l-\frac{1}{2}\rho+\frac{1}{2},\sigma\right),\ (a_p,A_p)\\ (b_q,B_q)\end{matrix}\right.\right]$$

$$=\frac{\Gamma\left(\frac{1}{2}+\frac{1}{2}\rho\right)}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k\left(\frac{1}{2}\rho\right)_k}{k!\left(\frac{1}{2}\right)_k}\ H^{n,m+2}_{p+2,q+1}\left[x\left|\begin{matrix}(-l,\sigma),\left(-l+\frac{1}{2},\sigma\right),(a_p,A_p)\\ (b_q,B_q),\ (-l+k,\sigma)\end{matrix}\right.\right]$$

where the conditions of validity for these special cases of interest are easily obtainable from their basic results.

(d) Further, if we make use of the relation (2.10) for developments of these results, we obtain the corresponding results in terms of Laguerre polynomials which may be required in various problems of Mathematical Physics.

Similar results can be recorded for (2.4) and from (2.11). Recently, Saxena, Bora, Kalla, Munot, Shah [3, 11, & 19], Batting and several other investigators have also contributed a lot of work on generalized Meijer's G-and Fox's H-functions of two variables. In numerous applied problems, application of such work is necessary and useful.

Since Fox's H-function of two variables is more general than even Meijer's

G-function of two variables, the results obtained here become master or key formulas from which a large number of known, new and interesting particular cases can be deduced for various functions appearing in theory of special functions.

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